LINEAR PARTIAL DIFFERENTIAL EQUATIONS AND FOURIER THEORY

Do you want a rigorous book that remembers where PDEs come from and what they look like? This highly visual introduction to linear PDEs and initial/boundary value problems connects the theory to physical reality, all the time providing a rigorous mathematical foundation for all solution methods.

Readers are gradually introduced to abstraction – the most powerful tool for solving problems – rather than simply drilled in the practice of imitating solutions to given examples. The book is therefore ideal for students in mathematics and physics who require a more theoretical treatment than is given in most introductory texts.

Also designed with lecturers in mind, the fully modular presentation is easily adapted to a course of one-hour lectures, and a suggested 12-week syllabus is included to aid planning. Downloadable files for the hundreds of figures, hundreds of challenging exercises, and practice problems that appear in the book are available online, as are solutions.

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To Joseph and Emma Pivato
for their support, encouragement,
and inspiring example.

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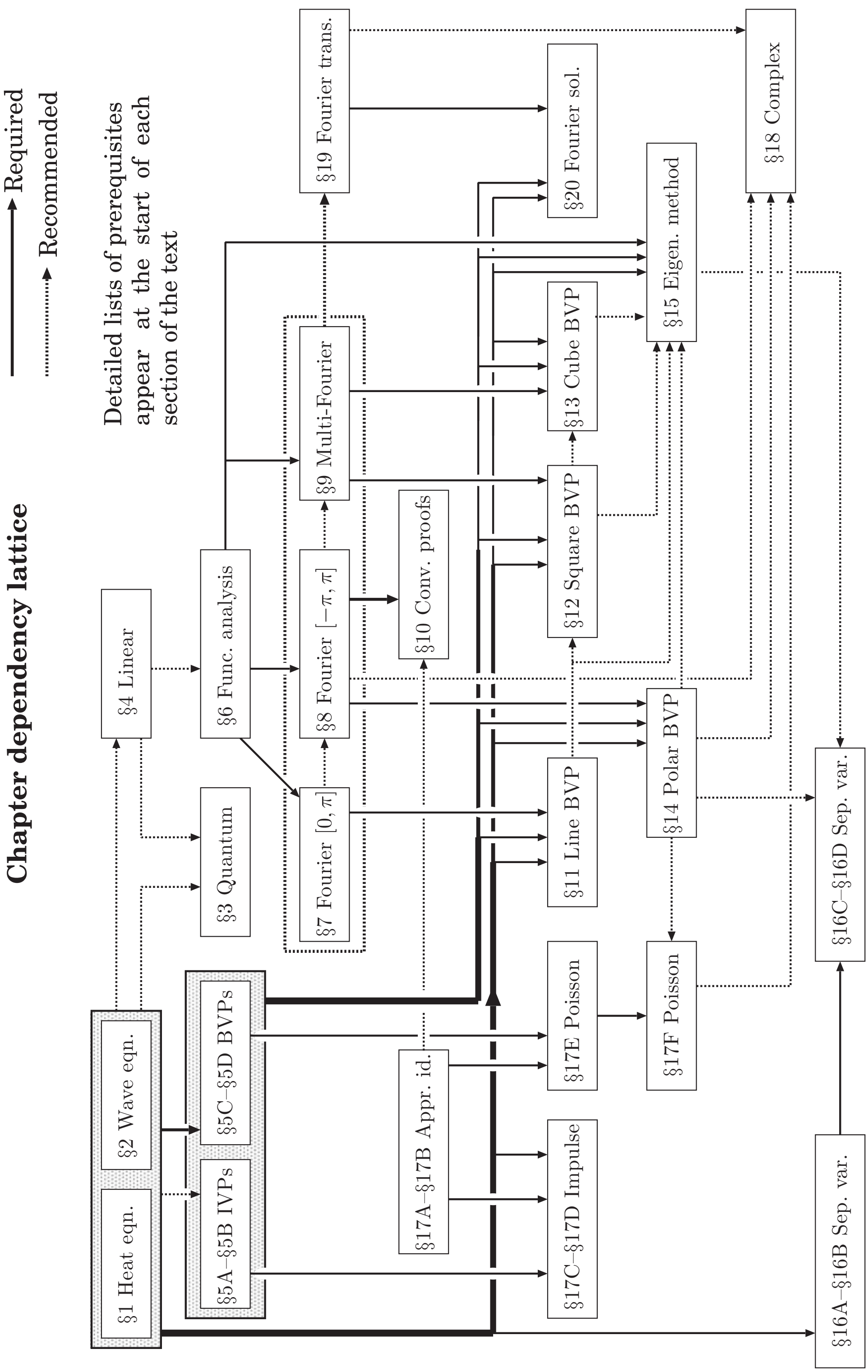
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Preface

This is a textbook for an introductory course on linear partial differential equations (PDEs) and initial/boundary value problems (I/BVPs). It also provides a mathematically rigorous introduction to Fourier analysis (Chapters 7, 8, 9, 10, and 19), which is the main tool used to solve linear PDEs in Cartesian coordinates. Finally, it introduces basic functional analysis (Chapter 6) and complex analysis (Chapter 18). The first is necessary to characterize rigorously the convergence of Fourier series, and also to discuss eigenfunctions for linear differential operators. The second provides powerful techniques to transform domains and compute integrals, and also offers additional insight into Fourier series.

This book is not intended to be comprehensive or encyclopaedic. It is designed for a one-semester course (i.e. 36–40 hours of lectures), and it is therefore strictly limited in scope. First, it deals mainly with *linear* PDEs with constant coefficients. Thus, there is no discussion of characteristics, conservation laws, shocks, variational techniques, or perturbation methods, which would be germane to other types of PDEs. Second, the book focuses mainly on concrete solution methods to specific PDEs (e.g. the Laplace, Poisson, heat, wave, and Schrödinger equations) on specific domains (e.g. line segments, boxes, disks, annuli, spheres), and spends rather little time on qualitative results about entire classes of PDEs (e.g. elliptic, parabolic, hyperbolic) on general domains. Only after a thorough exposition of these special cases does the book sketch the general theory; experience shows that this is far more pedagogically effective than presenting the general theory first. Finally, the book does not deal at all with numerical solutions or Galerkin methods.

One slightly unusual feature of this book is that, from the very beginning, it emphasizes the central role of eigenfunctions (of the Laplacian) in the solution methods for linear PDEs. Fourier series and Fourier–Bessel expansions are introduced as the orthogonal eigenfunction expansions which are most suitable in certain domains or coordinate systems. Separation of variables appears relatively late in the exposition (Chapter 16) as a convenient device to obtain such eigenfunctions.

The only techniques in the book which are not either implicitly or explicitly based on eigenfunction expansions are impulse-response functions and Green's functions (Chapter 17) and complex-analytic methods (Chapter 18).

Prerequisites and intended audience

This book is written for third-year undergraduate students in mathematics, physics, engineering, and other mathematical sciences. The only prererequisites are (1) *multivariate calculus* (i.e. partial derivatives, multivariate integration, changes of coordinate system) and (2) *linear algebra* (i.e. linear operators and their eigenvectors).

It might also be helpful for students to be familiar with the following: (1) the basic theory of ordinary differential equations (specifically, Laplace transforms, Frobenius method); (2) some elementary vector calculus (specifically, divergence and gradient operators); and (3) elementary physics (to understand the physical motivation behind many of the problems). However, none of these three things are really required.

In addition to this background knowledge, the book requires some ability at abstract mathematical reasoning. Unlike some 'applied math' texts, we do not suppress or handwave the mathematical theory behind the solution methods. At suitable moments, the exposition introduces concepts such as 'orthogonal basis', 'uniform convergence' vs. '$\mathbf{L}_2$-convergence', 'eigenfunction expansion', and 'self-adjoint operator'; thus, students must be intellectually capable of understanding abstract mathematical concepts of this nature. Likewise, the exposition is mainly organized in a 'definition $\to$ theorem $\to$ proof $\to$ example' format, rather than a 'problem $\to$ solution' format. Students must be able to understand abstract descriptions of general solution techniques, rather than simply learn by imitating worked solutions to special cases.

Conventions in the text

$*$ in the title of a chapter or section indicates 'optional' material which is not part of the core syllabus.

(Optional) in the margin indicates that a particular theorem or statement is 'optional' in the sense that it is not required later in the text.

Ⓔ in the margin indicates the location of an exercise. (Shorter exercises are sometimes embedded within the exposition.)

⧫ indicates the ends of more lengthy exercises.

□ ends the proof of a theorem.

◇ indicates the end of an example.

⋄ ends the proof of a 'claim' within the proof of a theorem.

△ ends the proof of a 'subclaim' within the proof of a claim.

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None of these people are responsible for any remaining errors, omissions, or other flaws in the book (of which there are no doubt many). If you find an error or some other deficiency in the book, please contact me at marcuspivato@trentu.ca.

This book would not have been possible without open source software. The book was prepared entirely on the LINUX operating system (initially REDHAT,[1] and later UBUNTU[2]). All the text is written in Leslie Lamport's LaTeX2e typesetting language,[3] and was authored using Richard Stallman's EMACS editor.[4] The illustrations were hand-drawn using William Chia-Wei Cheng's excellent TGIF object-oriented drawing program.[5] Additional image manipulation and post-processing was done with GNU IMAGE MANIPULATION PROGRAM (GIMP).[6] Many of the plots were created using GNUPLOT.[7,8] I would like to take this opportunity to thank the many people in the open source community who have developed this software.

Finally, and most importantly, I would like to thank my beloved wife and partner, Reem Yassawi, and our wonderful children, Leila and Aziza, for their support and for their patience with my many long absences.

[1] http://www.redhat.com. [2] http://www.ubuntu.com. [3] http://www.latex-project.org.
[4] http://www.gnu.org/software/emacs/emacs.html. [5] http://bourbon.usc.edu:8001/tgif.
[6] http://www.gimp.org. [7] http://www.gnuplot.info.
[8] Many other plots were generated using Waterloo MAPLE (http://www.maplesoft.com), which unfortunately is *not* open-source.

What's good about this book?

This text has many advantages over most other introductions to partial differential equations.

Illustrations

PDEs are physically motivated and geometrical objects; they describe curves, surfaces, and scalar fields with special geometric properties, and the way these entities evolve over time under endogenous dynamics. To understand PDEs and their solutions, it is necessary to visualize them. Algebraic formulae are just a language used to communicate such visual ideas in lieu of pictures, and they generally make a poor substitute. This book has over 300 high-quality illustrations, many of which are rendered in three dimensions. In the online version of the book, most of these illustrations appear in full colour. Also, the website contains many animations which do not appear in the printed book.

Most importantly, on the book website, all illustrations are *freely available* under a Creative Commons Attribution Noncommercial Share-Alike License.[1] This means that you are free to download, modify, and utilize the illustrations to prepare your own course materials (e.g. printed lecture notes or beamer presentations), as long as you attribute the original author. Please visit <http://xaravve.trentu.ca/pde>.

Physical motivation

Connecting the math to physical reality is critical: it keeps students motivated, and helps them interpret the mathematical formalism in terms of their physical intuitions about diffusion, vibration, electrostatics, etc. Chapter 1 of this book discusses the

[1] See http://creativecommons.org/licenses/by-nc-sa/3.0.

physics behind the heat, Laplace, and Poisson equations, and Chapter 2 discusses the wave equation. An unusual addition to this text is Chapter 3, which discusses quantum mechanics and the Schrödinger Equation (one of the major applications of PDE theory in modern physics).

Detailed syllabus

Difficult choices must be made when turning a 600+ page textbook into a feasible 12-week lesson plan. It is easy to run out of time or inadvertently miss something important. To facilitate this task, this book provides a lecture-by-lecture breakdown of how the author covers the material (see p. xxv). Of course, each instructor can diverge from this syllabus to suit the interests/background of their students, a longer/shorter teaching semester, or their personal taste. But the prefabricated syllabus provides a base to work from, and will save most instructors a lot of time and aggravation.

Explicit prerequisites for each chapter and section

To save time, an instructor might want to skip a certain chapter or section, but worries that it may end up being important later. We resolve this problem in two ways. First, p. xiv provides a '*chapter dependency lattice*', which summarizes the large-scale structure of logical dependencies between the chapters of the book. Second, every section of every chapter begins with an explicit list of 'required' and 'recommended' prerequisite sections; this provides more detailed information about the small-scale structure of logical dependencies between sections. By tracing backward through this 'lattice of dependencies', you can figure out exactly what background material you must cover to reach a particular goal. This makes the book especially suitable for self-study.

Flat dependency lattice

There are many 'paths' through the 20-chapter dependency lattice on p. xiv every one of which is only *seven* chapters or less in length. Thus, an instructor (or an autodidact) can design many possible syllabi, depending on their interests, and can quickly move to advanced material. The 'Suggested 12-week syllabus' on p. xxv describes a gentle progression through the material, covering most of the 'core' topics in a 12-week semester, emphasizing concrete examples and gradually escalating the abstraction level. The Chapter Dependency Lattice suggests some other possibilities for 'accelerated' syllabi focusing on different themes.

- *Solving PDEs with impulse response functions.* Chapters 1, 2, 5, and 17 only.
- *Solving PDEs with Fourier transforms.* Chapters 1, 2, 5, 19, and 20 only. (Pedagogically speaking, Chapters 8 and 9 will help the student understand Chapter 19, and Chapters 11–13 will help the student understand Chapter 20. Also, it is interesting to see how the 'impulse response' methods of Chapter 17 yield the same solutions as the 'Fourier methods' of Chapter 20, using a totally different approach. However, strictly speaking, none of Chapters 8–13 or 17 is logically necessary.)
- *Solving PDEs with separation of variables.* Chapters 1, 2, and, 16 only. (However, without at least Chapters 12, and 14, the ideas of Chapter 16 will seem somewhat artificial and pointless.)
- *Solving I/BVPs using eigenfunction expansions.* Chapters 1, 2, 4, 5, 6, and 15. (It would be pedagogically better to also cover Chapters 9 and 12, and probably Chapter 14. But, strictly speaking, none of these is logically necessary.)
- *Tools for quantum mechanics.* Section 1B, then Chapters 3, 4, 6, 9, 13, 19, and 20 (skipping material on Laplace, Poisson, and wave equations in Chapters 13 and 20, and adapting the solutions to the heat equation into solutions to the Schrödinger Equation).
- *Fourier theory.* Section 4A, then Chapters 6, 7, 8, 9, 10, and 19. Finally, Sections 18A, 18C, 18E, and 18F provide a 'complex' perspective. (Section 18H also contains some useful computational tools.)
- *Crash course in complex analysis.* Chapter 18 is logically independent of the rest of the book, and rigorously develops the main ideas in complex analysis from first principles. (However, the emphasis is on applications to PDEs and Fourier theory, so some of the material may seem esoteric or unmotivated if read in isolation from other chapters.)

Highly structured exposition, with clear motivation up front

The exposition is broken into small, semi-independent logical units, each of which is clearly labelled, and which has a clear purpose or meaning which is made immediately apparent. This simplifies the instructor's task; it is not necessary to spend time restructuring and summarizing the text material because it is already structured in a manner which self-summarizes. Instead, instructors can focus more on explanation, motivation, and clarification.

Many 'practice problems' (with complete solutions and source code available online)

Frequent evaluation is critical to reinforce material taught in class. This book provides an extensive supply of (generally simple) computational 'practice problems' at the end of each chapter. Completely worked solutions to virtually all of these problems are available on the book website. Also on the book website, the LaTeX source code for all problems and solutions is *freely available* under a Creative

Commons Attribution Noncommercial Share-Alike License.[2] Thus, an instructor can download and edit this source code, and easily create quizzes, assignments, and matching solutions for students.

Challenging exercises without solutions

Complex theoretical concepts cannot really be tested in quizzes, and do not lend themselves to canned 'practice problems'. For a more theoretical course with more mathematically sophisticated students, the instructor will want to assign some proof-related exercises for homework. This book has more than 420 such exercises scattered throughout the exposition; these are flagged by an 'Ⓔ' symbol in the margin, as shown here. Many of these exercises ask the student to prove a major result from the text (or a component thereof). This is the best kind of exercise, because it reinforces the material taught in class, and gives students a sense of ownership of the mathematics. Also, students find it more fun and exciting to prove important theorems, rather than solving esoteric make-work problems. Ⓔ

Appropriate rigour

The solutions of PDEs unfortunately involve many technicalities (e.g. different forms of convergence; derivatives of infinite function series, etc.). It is tempting to handwave and gloss over these technicalities, to avoid confusing students. But this kind of pedagogical dishonesty actually makes students *more* confused; they know something is fishy, but they can't tell quite what. Smarter students know they are being misled, and may lose respect for the book, the instructor, or even the whole subject.

In contrast, this book provides a rigorous mathematical foundation for all its solution methods. For example, Chapter 6 contains a careful explanation of L^2-spaces, the various forms of convergence for Fourier series, and the differences between them – including the 'pathologies' which can arise when one is careless about these issues. I adopt a 'triage' approach to proofs: the simplest proofs are left as exercises for the motivated student (often with a step-by-step breakdown of the best strategy). The most complex proofs I have omitted, but I provide multiple references to other recent texts. In between are those proofs which are challenging but still accessible; I provide detailed expositions of these proofs. Often, when the text contains several variants of the same theorem, I prove one variant in detail, and leave the other proofs as exercises.

[2] See http://creativecommons.org/licenses/by-nc-sa/3.0.

Appropriate abstraction

It is tempting to avoid abstractions (e.g. linear differential operators, eigenfunctions), and simply present *ad hoc* solutions to special cases. This cheats the student. The right abstractions provide simple, yet powerful, tools that help students understand a myriad of seemingly disparate special cases within a single unifying framework. This book provides students with the opportunity to learn an abstract perspective once they are ready for it. Some abstractions are introduced in the main exposition, others are in optional sections, or in the philosophical preambles which begin each major part of the book.

Gradual abstraction

Learning proceeds from the concrete to the abstract. Thus, the book begins each topic with a specific example or a low-dimensional formulation, and only later proceeds to a more general/abstract idea. This introduces a lot of 'redundancy' into the text, in the sense that later formulations subsume the earlier ones. So the exposition is not as 'efficient' as it could be. This is a good thing. Efficiency makes for good reference books, but lousy texts.

For example, when introducing the heat equation, Laplace equation, and wave equation in Chapters 1 and 2, I first derive and explain the one-dimensional version of each equation, then the two-dimensional version, and, finally, the general, D-dimensional version. Likewise, when developing the solution methods for BVPs in Cartesian coordinates (Chapters 11–13), I confine the exposition to the interval $[0, \pi]$, the square $[0, \pi]^2$, and the cube $[0, \pi]^3$, and assume all the coefficients in the differential equations are unity. Then the exercises ask the student to state and prove the appropriate generalization of each solution method for an interval/rectangle/box of arbitrary dimensions, and for equations with arbitrary coefficients. The general method for solving I/BVPs using eigenfunction expansions only appears in Chapter 15, after many special cases of this method have been thoroughly exposited in Cartesian and polar coordinates (Chapters 11–14).

Likewise, the development of Fourier theory proceeds in gradually escalating levels of abstraction. First we encounter Fourier (co)sine series on the interval $[0, \pi]$ (§7A); then on the interval $[0, L]$ for arbitrary $L > 0$ (§7B). Then Chapter 8 introduces 'real' Fourier series (i.e. with both sine and cosine terms), and then complex Fourier series (§8D). Then, Chapter 9 introduces two-dimensional (co)sine series and, finally, D-dimensional (co)sine series.

Expositional clarity

Computer scientists have long known that it is easy to write software that *works*, but it is much more difficult (and important) to write working software that *other people* can understand. Similarly, it is relatively easy to write formally correct mathematics; the real challenge is to make the mathematics easy to read. To achieve this, I use several techniques. I divide proofs into semi-independent modules ('claims'), each of which performs a simple, clearly defined task. I integrate these modules together in an explicit hierarchical structure (with 'subclaims' inside of 'claims'), so that their functional interdependence is clear from visual inspection. I also explain formal steps with parenthetical heuristic remarks. For example, in a long string of (in)equalities, I often attach footnotes to each step, as follows:

' $A \underset{(*)}{=} B \underset{(\dagger)}{\leq} C \underset{(\ddagger)}{<} D$. Here, (∗) is because [. . .]; (†) follows from [. . .], and (‡) is because [. . .].'

Finally, I use letters from the same 'lexicographical family' to denote objects which 'belong' together. For example: If $\mathcal{S}$ and $\mathcal{T}$ are sets, then elements of $\mathcal{S}$ should be $s_1, s_2, s_3, \ldots$, while elements of $\mathcal{T}$ are $t_1, t_2, t_3, \ldots$. If $\mathbf{v}$ is a vector, then its entries should be $v_1, \ldots, v_N$. If $\mathbf{A}$ is a matrix, then its entries should be $a_{11}, \ldots, a_{NM}$. I reserve upper-case letters (e.g. $J, K, L, M, N, \ldots$) for the bounds of intervals or indexing sets, and then use the corresponding lower-case letters (e.g. $j, k, l, m, n, \ldots$) as indexes. For example, $\forall n \in \{1, 2, \ldots, N\}$, $A_n := \sum_{j=1}^{J} \sum_{k=1}^{K} a_{jk}^{n}$.

Clear and explicit statements of solution techniques

Many PDE texts contain very few theorems; instead they try to develop the theory through a long sequence of worked examples, hoping that students will 'learn by imitation', and somehow absorb the important ideas 'by osmosis'. However, less gifted students often just imitate these worked examples in a slavish and uncomprehending way. Meanwhile, the more gifted students do not want to learn 'by osmosis'; they want clear and precise statements of the main ideas.

The problem is that most solution methods in PDEs, if stated as theorems in full generality, are incomprehensible to many students (especially the non-math majors). My solution is this: I provide explicit and precise statements of the solution method for almost every possible combination of (1) several major PDEs, (2) several kinds of boundary conditions, and (3) several different domains. I state these solutions as *theorems*, not as 'worked examples'. I follow each of these theorems with several completely worked examples. Some theorems I prove, but most of the proofs are left as exercises (often with step-by-step hints).

Of course, this approach is highly redundant, because I end up stating more than 20 theorems, which really are all special cases of three or four general results (for example, the general method for solving the heat equation on a compact domain with Dirichlet boundary conditions, using an eigenfunction expansion). However, this sort of redundancy is *good* in an elementary exposition. Highly 'efficient' expositions are pleasing to our aesthetic sensibilities, but they are dreadful for pedagogical purposes.

However, one must not leave the students with the impression that the theory of PDEs is a disjointed collection of special cases. To link together all the 'homogeneous Dirichlet heat equation' theorems, for example, I explicitly point out that they all utilize the same underlying strategy. Also, when a proof of one variant is left as an exercise, I encourage students to imitate the (provided) proofs of previous variants. When the students understand the underlying similarity between the various special cases, *then* it is appropriate to state the general solution. The students will almost feel they have figured it out for themselves, which is the best way to learn something.

Suggested 12-week syllabus

Week 1: *Heat and diffusion-related PDEs*

- **Lecture 1:** Appendix A–Appendix E *Review of multivariate calculus; introduction to complex numbers.*
- **Lecture 2:** §1A–§1B *Fourier's law; the heat equation.*
- **Lecture 3:** §1C–§1D *Laplace equation; Poisson equation.*

Week 2: *Wave-related PDEs; quantum mechanics*

- **Lecture 1:** §1E; §2A *Properties of harmonic functions; spherical means.*
- **Lecture 2:** §2B–§2C *Wave equation; telegraph equation.*
- **Lecture 3:** Chapter 3 *The Schrödinger equation in quantum mechanics.*

Week 3: *General theory*

- **Lecture 1:** §4A–§4C *Linear PDEs: homogeneous vs. nonhomogeneous.*
- **Lecture 2:** §5A; §5B *Evolution equations and initial value problems.*
- **Lecture 3:** §5C *Boundary conditions and boundary value problems.*

Week 4: *Background to Fourier theory*

- **Lecture 1:** §5D *Uniqueness of solutions to BVPs*; §6A *inner products.*
- **Lecture 2:** §6B–§6D *L^2-space; orthogonality.*
- **Lecture 3:** §6E(i)–(iii) *L^2-convergence; pointwise convergence; uniform convergence.*

Week 5: *One-dimensional Fourier series*

- **Lecture 1:** §6E(iv) *Infinite series*; §6F *orthogonal bases*; §7A *Fourier (co/sine) series: definition and examples.*
- **Lecture 2:** §7C(i)–(v) *Computing Fourier series of polynomials, piecewise linear and step functions.*
- **Lecture 3:** §11A–§11C *Solution to heat equation and Poisson equation on a line segment.*

Week 6: *Fourier solutions for BVPs in one and two dimensions*

- **Lecture 1:** §11B–§12A *Wave equation on line segment and Laplace equation on a square.*
- **Lecture 2:** §9A–§9B *Multidimensional Fourier series.*
- **Lecture 3:** §12B–§12C(i) *Solution to heat equation and Poisson equation on a square.*

Week 7: *Fourier solutions for two-dimensional BVPs in Cartesian and polar coordinates*

- **Lecture 1:** §12C(ii), §12D *Solution to Poisson equation and wave equation on a square.*
- **Lecture 2:** §5C(iv); §8A–§8B *Periodic boundary conditions; real Fourier series.*
- **Lecture 3:** §14A; §14B(i)–(iv) *Laplacian in polar coordinates; Laplace equation on (co)disk.*

Week 8: *BVPs in polar coordinates; Bessel functions*

- **Lecture 1:** §14C *Bessel functions.*
- **Lecture 2:** §14D–§14F *Heat, Poisson, and wave equations in polar coordinates.*
- **Lecture 3:** §14G *Solving Bessel's equation with the method of Frobenius.*

Week 9: *Eigenbases; separation of variables*

- **Lecture 1:** §15A–§15B *Eigenfunction solutions to BVPs.*
- **Lecture 2:** §15B; §16A–§16B *Harmonic bases; separation of variables in Cartesian coordinates.*
- **Lecture 3:** §16C–§16D *Separation of variables in polar and spherical coordinates; Legendre polynomials.*

Week 10: *Impulse response methods*

- **Lecture 1:** §17A–§17C *Impulse response functions; convolution; approximations of identity; Gaussian convolution solution for heat equation.*
- **Lecture 2:** §17C–§17F *Gaussian convolutions continued; Poisson's solutions to Dirichlet problem on a half-plane and a disk.*
- **Lecture 3:** §14B(v); §17D *Poisson solution on disk via polar coordinates; d'Alembert solution to wave equation.*

Week 11: *Fourier transforms*

- **Lecture 1:** §19A *One-dimensional Fourier transforms.*
- **Lecture 2:** §19B *Properties of one-dimensional Fourier transform.*
- **Lecture 3:** §20A; §20C *Fourier transform solution to heat equation; Dirchlet problem on half-plane.*

Week 12: *Fourier transform solutions to PDEs*

- **Lecture 1:** §19D, §20B(i) *Multidimensional Fourier transforms; solution to wave equation.*

Lecture 2: §20B(ii); §20E *Poisson's spherical mean solution; Huygen's principle; the general method.*

Lecture 3: (Time permitting) §19G or §19H *Heisenberg uncertainty or Laplace transforms*.

In a longer semester or a faster paced course, one could also cover parts of Chapter 10 (*Proofs of Fourier convergence*) and/or Chapter 18 (*Applications of complex analysis*).