

## MODULES OVER ENDOMORPHISM RINGS

This is an extensive synthesis of recent work in the study of endomorphism rings and their modules, bringing together direct sum decompositions of modules, the class number of an algebraic number field, point set topological spaces, and classical noncommutative localization.

The main idea behind the book is to study modules  $G$  over a ring  $R$  via their endomorphism ring  $\text{End}_R(G)$ . The author discusses a wealth of results that classify  $G$  and  $\text{End}_R(G)$  via numerous properties, and in particular results from point set topology are used to provide a complete characterization of the direct sum decomposition properties of  $G$ .

For graduate students this is a useful introduction, while the more experienced mathematician will discover that the book contains results that are not otherwise available. Each chapter contains a list of exercises and problems for future research, which provide a springboard for students entering modern professional mathematics.

THEODORE G. FATICONI is Professor in the Mathematics Department at Fordham University, New York.

---

 ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS
 

---

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit

<http://www.cambridge.org/uk/series/sSeries.asp?code=EOM>

- 70 A. Pietsch and J. Wenzel *Orthonormal Systems and Banach Space Geometry*
- 71 G. E. Andrews, R. Askey and R. Roy *Special Functions*
- 72 R. Ticciati *Quantum Field Theory for Mathematicians*
- 73 M. Stern *Seminodular Lattices*
- 74 I. Lasiecka and R. Triggiani *Control Theory for Partial Differential Equations I*
- 75 I. Lasiecka and R. Triggiani *Control Theory for Partial Differential Equations II*
- 76 A. A. Ivanov *Geometry of Sporadic Groups I*
- 77 A. Schinzel *Polynomials with Special Regard to Reducibility*
- 78 T. Beth, D. Jungnickel and H. Lenz *Design Theory II, 2nd edn*
- 79 T. W. Palmer *Banach Algebras and the General Theory of \*-Algebras II*
- 80 O. Storkmark *Lie's Structural Approach to PDE Systems*
- 81 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables*
- 82 J. P. Mayberry *The Foundations of Mathematics in the Theory of Sets*
- 83 C. Foias, O. Manley, R. Rosa and R. Temam *Navier–Stokes Equations and Turbulence*
- 84 B. Polster and G. Steinke *Geometries on Surfaces*
- 85 R. B. Paris and D. Kaminski *Asymptotics and Mellin–Barnes Integrals*
- 86 R. McEliece *The Theory of Information and Coding, 2nd edn*
- 87 B. A. Magurn *An Algebraic Introduction to K-Theory*
- 88 T. Mora *Solving Polynomial Equation Systems I*
- 89 K. Bichteler *Stochastic Integration with Jumps*
- 90 M. Lothaire *Algebraic Combinatorics on Words*
- 91 A. A. Ivanov and S. V. Shpectorov *Geometry of Sporadic Groups II*
- 92 P. McMullen and E. Schulte *Abstract Regular Polytopes*
- 93 G. Gierz *et al. Continuous Lattices and Domains*
- 94 S. R. Finch *Mathematical Constants*
- 95 Y. Jabri *The Mountain Pass Theorem*
- 96 G. Gasper and M. Rahman *Basic Hypergeometric Series, 2nd edn*
- 97 M. C. Pedicchio and W. Tholen (eds.) *Categorical Foundations*
- 98 M. E. H. Ismail *Classical and Quantum Orthogonal Polynomials in One Variable*
- 99 T. Mora *Solving Polynomial Equation Systems II*
- 100 E. Olivieri and M. Eulália Vares *Large Deviations and Metastability*
- 101 A. Kushner, V. Lychagin and V. Rubtsov *Contact Geometry and Nonlinear Differential Equations*
- 102 L. W. Beineke and R. J. Wilson (eds.) with P. J. Cameron *Topics in Algebraic Graph Theory*
- 103 O. Staffans *Well-Posed Linear Systems*
- 104 J. M. Lewis, S. Lakshmivarahan and S. K. Dhall *Dynamic Data Assimilation*
- 105 M. Lothaire *Applied Combinatorics on Words*
- 106 A. Markoe *Analytic Tomography*
- 107 P. A. Martin *Multiple Scattering*
- 108 R. A. Brualdi *Combinatorial Matrix Classes*
- 109 J. M. Borwein and J. D. Vanderwerff *Convex Functions*
- 110 M.-J. Lai and L. L. Schumaker *Spline Functions on Triangulations*
- 111 R. T. Curtis *Symmetric Generation of Groups*
- 112 H. Salzmann, T. Grundhöfer, H. Hähl and R. Löwen *The Classical Fields*
- 113 S. Peszat and J. Zabczyk *Stochastic Partial Differential Equations with Lévy Noise*
- 114 J. Beck *Combinatorial Games*
- 115 L. Barreira and Y. Pesin *Nonuniform Hyperbolicity*
- 116 D. Z. Arov and H. Dym *J-Contractive Matrix Valued Functions and Related Topics*
- 117 R. Glowinski, J.-L. Lions and J. He *Exact and Approximate Controllability for Distributed Parameter Systems*
- 118 A. A. Borovkov and K. A. Borovkov *Asymptotic Analysis of Random Walks*
- 119 M. Deza and M. Dutour Sikirić *Geometry of Chemical Graphs*
- 120 T. Nishiura *Absolute Measurable Spaces*
- 121 M. Prest *Purity, Spectra and Localisation*
- 122 S. Khrushchev *Orthogonal Polynomials and Continued Fractions: From Euler's Point of View*
- 123 H. Nagamochi and T. Ibaraki *Algorithmic Aspects of Graph Connectivity*
- 124 F. W. King *Hilbert Transforms I*
- 125 F. W. King *Hilbert Transforms II*
- 126 O. Calin and D.-C. Chang *Sub-Riemannian Geometry*
- 127 M. Grabisch, J.-L. Marichal, R. Mesiar and E. Pap *Aggregation Functions*
- 128 L. W. Beineke and R. J. Wilson (eds.) with J. L. Gross and T. W. Tucker *Topics in Topological Graph Theory*
- 129 J. Berstel, D. Perrin and C. Reutenauer *Codes and Automata*

# Modules over Endomorphism Rings

THEODORE G. FATICONI

*Fordham University*



CAMBRIDGE  
UNIVERSITY PRESS

Cambridge University Press  
978-0-521-19960-5 — Modules over Endomorphism Rings  
Theodore G. Faticoni  
Frontmatter  
[More Information](#)

**CAMBRIDGE**  
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom  
One Liberty Plaza, 20th Floor, New York, NY 10006, USA  
477 Williamstown Road, Port Melbourne, VIC 3207, Australia  
314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India  
103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.  
It furthers the University's mission by disseminating knowledge in the pursuit of  
education, learning and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)  
Information on this title: [www.cambridge.org/9780521199605](http://www.cambridge.org/9780521199605)

© T. G. Faticoni 2010

This publication is in copyright. Subject to statutory exception  
and to the provisions of relevant collective licensing agreements,  
no reproduction of any part may take place without the written  
permission of Cambridge University Press.

First published 2010

*A catalogue record for this publication is available from the British Library*

ISBN 978-0-521-19960-5 Hardback

Cambridge University Press has no responsibility for the persistence or  
accuracy of URLs for external or third-party internet websites referred to in  
this publication, and does not guarantee that any content on such websites is,  
or will remain, accurate or appropriate.

Cambridge University Press  
978-0-521-19960-5 — Modules over Endomorphism Rings  
Theodore G. Faticoni  
Frontmatter  
[More Information](#)

---

*To my wife Barbara Jean  
who helped me read our book of life.*

## Contents

---

<i>Preface</i>	<i>page</i> xiii
<b>1 Preliminary results</b>	<b>1</b>
1.1 Rings, modules, and functors	1
1.2 Azumaya–Krull–Schmidt theorem	3
1.3 The structure of rings	4
1.4 The Arnold–Lady theorem	5
<b>2 Class number of an abelian group</b>	<b>9</b>
2.1 Preliminaries	9
2.2 A functorial bijection	14
2.3 Internal cancellation	17
2.4 Power cancellation	19
2.5 Unique decomposition	21
2.6 Algebraic number fields	23
2.7 Exercises	25
2.8 Problems for future research	25
<b>3 Mayer–Vietoris sequences</b>	<b>26</b>
3.1 The sequence of groups	26
3.2 Analytic methods	32
3.3 Exercises	36
3.4 Problems for future research	36
<b>4 Lifting units</b>	<b>37</b>
4.1 Units and sequences	37
4.2 Calculations with primary ideals	40
4.3 Quadratic number fields	44
4.4 The Gaussian integers	45
4.5 Imaginary quadratic number fields	46
4.6 Exercises	48
4.7 Problems for future research	48
<b>5 The conductor</b>	<b>49</b>
5.1 Introduction	49
5.2 Some functors	51
5.3 Conductor of an <i>rtffr</i> ring	53
5.4 Local correspondence	54

viii	<i>Contents</i>	
	5.5 Exercises	60
	5.6 Problems for future research	60
<b>6</b>	<b>Conductors and groups</b>	<b>61</b>
	6.1 <i>Rtffr</i> groups	61
	6.2 Direct sum decompositions	63
	6.3 Locally semi-perfect rings	67
	6.4 Balanced semi-primary groups	70
	6.5 Examples	72
	6.6 Exercises	75
	6.7 Problems for future research	76
<b>7</b>	<b>Invertible fractional ideals</b>	<b>77</b>
	7.1 Introduction	77
	7.2 Functors and bijections	79
	7.3 The square	81
	7.4 Isomorphism classes	90
	7.5 The equivalence class $\{I\}$	91
	7.6 Commutative domains	93
	7.7 Cardinality of the kernels	95
	7.8 Relatively prime to $\tau$	96
	7.9 Power cancellation	98
	7.10 Algebraic number fields	100
	7.11 Exercises	102
	7.12 Problems for future research	102
<b>8</b>	<b><math>\mathcal{L}</math>-groups</b>	<b>103</b>
	8.1 $\mathcal{J}$ -groups, $\mathcal{L}$ -groups, and $\mathcal{S}$ -groups	103
	8.2 Eichler groups	103
	8.3 Direct sums of $\mathcal{L}$ -groups	106
	8.4 Eichler $\mathcal{L}$ -groups are $\mathcal{J}$ -groups	109
	8.5 Exercises	111
	8.6 Problems for future research	112
<b>9</b>	<b>Modules and homotopy classes</b>	<b>113</b>
	9.1 Right endomorphism modules	113
	9.1.1 The homotopy of $G$ -plexes	113
	9.1.2 Homotopy and homology	118
	9.1.3 Endomorphism modules as $G$ -plexes	120
	9.2 Two commutative triangles	128
	9.2.1 $G$ -Solvable $R$ -modules	129
	9.2.2 A factorization of the tensor functor	131
	9.3 Left endomorphism modules	134
	9.3.1 Duality	137
	9.4 Self-small self-slender modules	142
	9.5 $(\mu)$ Implies slender injectives	143
	9.6 Exercises	144
	9.7 Problems for future research	146

## Contents

ix

<b>10</b>	<b>Tensor functor equivalences</b>	<b>148</b>
10.1	Small projective generators	148
10.2	Quasi-projective modules	153
10.3	Flat endomorphism modules	157
10.3.1	A category equivalence for submodules of free modules	157
10.3.2	Right ideals in endomorphism rings	161
10.3.3	A criterion for $E$ -flatness	162
10.4	Orsatti and Menini's $*$ -modules	163
10.5	Dualities from injective properties	166
10.5.1	$G$ -Cosolvable $R$ -modules	167
10.5.2	A factorization of $\text{Hom}_R(\cdot, G)$	168
10.5.3	Dualities for the dual functor	169
10.6	Exercises	171
10.7	Problems for future research	173
<b>11</b>	<b>Characterizing endomorphisms</b>	<b>175</b>
11.1	Flat endomorphism modules	175
11.2	Homological dimension	177
11.2.1	Definitions and examples	177
11.2.2	The exact dimension of a $G$ -plex	179
11.2.3	The projective dimension of a $G$ -plex	180
11.3	The flat dimension	184
11.4	Global dimensions	188
11.5	Small global dimensions	191
11.5.1	Baer's lemma	191
11.5.2	Semi-simple rings	194
11.5.3	Right hereditary rings	195
11.5.4	Global dimension at most 3	201
11.6	Injective dimensions and modules	202
11.6.1	A review of $G$ -cplexes	202
11.6.2	Injective endomorphism rings	206
11.6.3	Left homological dimensions	209
11.7	A glossary of terms	212
11.8	Exercises	214
11.9	Problems for future research	218
<b>12</b>	<b>Projective modules</b>	<b>219</b>
12.1	Projectives	219
12.2	Finitely generated modules	223
12.3	Exercises	228
12.4	Problems for future research	228
<b>13</b>	<b>Finitely generated modules</b>	<b>229</b>
13.1	Beaumont–Pierce	229
13.2	Noetherian modules	235
13.3	Generators	237



13.4	Exercises	241
13.5	Problems for future research	241
<b>14</b>	<b><i>Rtffr</i> <math>E</math>-projective groups</b>	<b>242</b>
14.1	Introduction	242
14.2	The UConn '81 Theorem	245
14.3	Exercises	248
14.4	Problems for future research	248
<b>15</b>	<b>Injective endomorphism modules</b>	<b>249</b>
15.1	$G$ -Monomorphisms	249
15.2	Injective properties	251
15.3	$G$ -Cogenerators	258
15.4	Projective modules revisited	261
15.5	Examples	262
15.6	Exercises	263
15.7	Problems for future research	264
<b>16</b>	<b>A diagram of categories</b>	<b>265</b>
16.1	The diagram	265
16.2	Smallness and slenderness	269
16.3	Coherent objects	272
16.4	The construction function	272
16.5	The Greek maps	274
16.6	Applications	275
16.6.1	Complete sets of invariants	275
16.6.2	Unique topological decompositions	276
16.6.3	Homological dimensions	279
16.7	Exercises	282
16.8	Problems for future research	282
<b>17</b>	<b>Diagrams of abelian groups</b>	<b>284</b>
17.1	The ring $\text{End}_{\mathbb{C}}(X)$	285
17.2	Topological complexes	286
17.3	Categories of complexes	288
17.4	Commutative triangles	291
17.5	Three diamonds	294
17.5.1	A diagram for an abelian groups	294
17.5.2	Self-small and self-slender	296
17.5.3	Coherent complexes	298
17.6	Prism diagrams	300
17.7	Direct sums	301
17.8	Algebraic number fields	304
17.9	Exercises	309
17.10	Problems for future research	311
<b>18</b>	<b>Marginal isomorphisms</b>	<b>312</b>
18.1	Ore localization	312
18.1.1	Preliminary concepts and examples	313
18.1.2	Noncommutative localization	315

<i>Contents</i>		xi
18.2	Marginal isomorphisms	322
18.2.1	Margimorphism and localizations	323
18.2.2	Marginal summands	327
18.2.3	Marginal summands as projectives	329
18.2.4	Projective $Q_G$ -modules	331
18.3	Uniqueness of direct summands	333
18.3.1	Totally indecomposable modules	333
18.3.2	Morphisms of totally indecomposables	334
18.3.3	Semi-simple marginal summands	337
18.3.4	Jónsson's theorem and margimorphisms	339
18.4	Nilpotent sets and margimorphism	342
18.5	Isomorphism from margimorphism	346
18.6	Semi-simple endomorphism rings	353
18.7	Exercises	358
18.8	Problems for future research	360
	<i>Bibliography</i>	362
	<i>Index</i>	368

## Preface

---

The chapters in this book are from papers published or submitted to peer-reviewed journals. These papers were written by the author during the calendar years 2006–2008.

There is a simple example that motivates the point of view of this text. Let  $\mathbf{k}$  be a field, let  $V$  be an  $n$ -dimensional  $\mathbf{k}$ -vector space for some integer  $n > 0$ , and let  $E = \text{Mat}_n(\mathbf{k})$  denote the ring of  $n \times n$ -matrices over  $\mathbf{k}$ . Fix an ordered basis  $\beta$  for  $V$  and let  $[\mathbf{v}]_\beta$  denote the vector representation for  $\mathbf{v}$  relative to  $\beta$ . Given  $r \in E$  and  $\mathbf{v} \in V$  then we define

$$r\mathbf{v} = r \cdot [\mathbf{v}]_\beta,$$

where  $\cdot$  is the usual multiplication between the  $n \times n$  matrix  $r$  and the column vector  $[\mathbf{v}]_\beta$ . This multiplication makes  $V$  a left  $E$ -module. Given a right ideal  $I \subset E$  we define

$$IV = \left\{ \sum_i r_i \mathbf{v}_i \mid \text{finitely many elements } r_i \in I \text{ and } \mathbf{v}_i \in V \right\}.$$

Then the assignment

$$I \longmapsto IV$$

defines a bijection between the set of right ideals of  $E$  and the set of  $\mathbf{k}$ -subspaces of  $V$ . Thus we can study some properties of  $V$  by studying the right ideals in the ring  $E$ . Notice that we have passed from a strictly additive setting into a setting that is additive and multiplicative. This gain in structure improves our chances of solving certain problems concerning  $V$ .

There is little hope of generalizing the bijection  $I \longmapsto IV$  to more general modules over associative rings without sacrificing something, so we hope for the best possible generalization. To find this generalization we will use elements from ring theory, module theory, and some elementary homology and homotopy theory of complexes over associative rings, and in at least a couple of instances we use some point set topology. More details follow.

Most modern research into direct sum decompositions of reduced torsion-free finite rank abelian groups (now called *rtffr groups*) begins with a study of projective modules over  $\text{End}(G)$ . This stems from the Arnold–Lady theorem, which shows that direct summands of  $G^n$  correspond to projective direct summands of  $\text{End}(G)^n$ .

This method begins a study of modules  $G$  over a ring  $R$  and the endomorphism ring  $\text{End}_R(G)$ . We begin by constructing a category  $G$ -plex of what are called  $G$ -plexes. This category is category equivalent to  $\mathbf{Mod}\text{-}\text{End}_R(G)$ , which makes  $G$ -plex the category to be studied if we wish to characterize  $G$  or  $\text{End}_R(G)$ . A duality is used to characterize  $\text{End}_R(G)\text{-Mod}$  in terms of a category  $G$ -coplex. Thus we can characterize the rings  $\text{End}_R(G)$  whose properties are on the following list of properties of rings:

1. right or left hereditary
2. right or left Noetherian
3. right or left coherent
4. right or left FP-injective
5. right or left self-injective
6. right or left cogenerator
7. right PF rings
8. QF rings

We consider  $\text{End}_R(G)$  and the left  $\text{End}_R(G)$ -module  $G$ , and we characterize several integers associated with rings and modules. Specifically we characterize the integers on the following list:

1. projective dimension of  $G$
2. injective dimension of  $G$
3. flat dimension of  $G$
4. right or left global dimension of  $\text{End}_R(G)$

Several properties are left as exercises.

One of the purposes of this book is to show that we can study groups locally isomorphic to  $G$  by studying invertible fractional right ideals of  $E(G)$  where

$$E(G) = \text{End}(G)/\mathcal{N}(\text{End}(G)).$$

For example, let  $n > 0$  be an integer, and given a commutative prime ring  $R$ , let  $\text{Pic}(R)$  denote the abelian group of isomorphism classes of invertible fractional right ideals of  $R$ . If  $G$  is a strongly indecomposable *rtffr* group and if  $E(G)$  is commutative then the set of isomorphism classes ( $H$ ) of groups  $H$  that are *locally isomorphic* to  $G^n$  is bijective with the finite abelian group  $\text{Pic}(E(G))$ . In this setting we show that if  $H$ ,  $K$ , and  $L$  are direct summands of  $G^n$ , then cancellation in the isomorphism  $H \oplus K \cong H \oplus L$  can be viewed as cancellation of elements in the abelian group  $\text{Pic}(E(G))$ .

This point of view gives a new insight into the problem of finding the class number  $h(\mathbf{k})$  of the algebraic number field  $\mathbf{k}$ . For example, those  $\mathbf{k}$  with  $h(\mathbf{k}) = 1$  are classified

in the following result. Let  $\bar{E}$  denote the algebraic integers in  $\mathbf{k}$ . Let  $\Omega(\bar{E}) = \{\text{rtffr groups } G \mid \text{End}(G) \cong \bar{E}\}$ .

**Theorem.** *The following are equivalent for the algebraic number field  $\mathbf{k}$ .*

1.  $h(\mathbf{k}) = 1$ .
2. Each  $G \in \Omega(\mathbf{k})$  has the power cancellation property. (For each integer  $m > 0$  and group  $H$ ,  $G^m \cong H^m$  implies that  $G \cong H$ .)
3. Each group  $H$  that is locally isomorphic to  $G$  is isomorphic to  $G$ .

Some research over the thirty-year period from 1970 to 2000 dealt with the rtffr groups  $G$  that were finitely generated left  $\text{End}(G)$ -modules, or projective left  $\text{End}(G)$ -modules, or that had right hereditary endomorphism ring. Our thirty-odd pages on this type of result give us a unified approach to these problems and extends existing results. Subsequently, we use the machinery developed in Chapter 9 to characterize the left  $\text{End}(G)$ -module  $G$  that possesses some properties from the following list:

1. finitely generated
2. finitely presented
3. coherent
4. projective
5. quasi-projective
6. possesses a projective cover
7. cogenerator
8. generator
9. progenerator
10. quasi-progenerator
11. Noetherian

Let  $R$  be an associative ring with identity, let  $G$  be a right  $R$ -module, and let

$$\text{End}_R(G)$$

denote the ring of  $R$ -endomorphisms of  $G$ . The module  $G$  is *self-small* if for each index set  $\mathcal{I}$  and each  $R$ -module map  $\phi : G \rightarrow G^{(\mathcal{I})}$  there is a finite set  $\mathcal{J} \subset \mathcal{I}$  such that  $\phi(G) \subset G^{(\mathcal{J})}$ . In other words there is a natural isomorphism

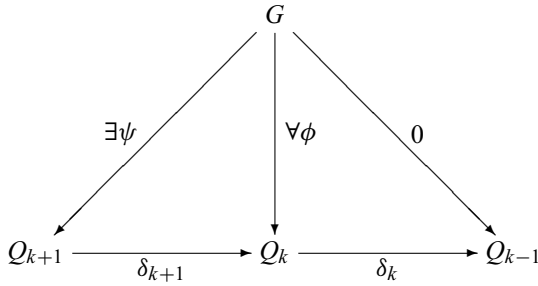
$$\text{Hom}_R(G, G)^{(\mathcal{I})} \longrightarrow \text{Hom}_R(G, G^{(\mathcal{I})}).$$

Let  $\mathbf{P}(G) = \{\text{right } R\text{-modules } Q \mid Q \oplus Q' \cong G^{(\mathcal{I})} \text{ for some index set } \mathcal{I} \text{ and some right } R\text{-module } Q'\}$ . A  $G$ -plex is a complex

$$Q = \cdots \xrightarrow{\delta_3} Q_2 \xrightarrow{\delta_2} Q_1 \xrightarrow{\delta_1} Q_0$$

with the properties that

1.  $Q_k \in \mathbf{P}(G)$  for each  $k \geq 0$  and
2.  $G$  has the following lifting property for each  $k \geq 1$ . Given a map  $\phi : G \rightarrow Q_k$  such that  $\delta_k \phi = 0$  there is a map  $\psi : G \rightarrow Q_{k+1}$  such that  $\phi = \delta_{k+1} \psi$  as in the commutative triangle



of right  $R$ -modules.

The category of  $G$ -plexes  $G\text{-Plex}$  is the additive category whose objects are the  $G$ -plexes  $\mathcal{Q}$  and whose morphisms are homotopy equivalence classes  $[f]$  of chain maps

$$f : \mathcal{Q} \rightarrow \mathcal{Q}'$$

between  $G$ -plexes  $\mathcal{Q}$  and  $\mathcal{Q}'$ .

If we let  $\mathbf{Mod}\text{-End}_R(G)$  denote the category of right  $\text{End}_R(G)$ -modules then the functor

$$h_G(\cdot) : G\text{-Plex} \rightarrow \mathbf{Mod}\text{-End}_R(G)$$

sends  $\mathcal{Q} \in G\text{-Plex}$  to the zeroth homology group of the complex

$$\text{Hom}(G, \mathcal{Q}) = \cdots \xrightarrow{\delta_2^*} \text{Hom}_R(G, Q_1) \xrightarrow{\delta_1^*} \text{Hom}_R(G, Q_0),$$

or in other words

$$h_G(\mathcal{Q}) = \text{coker } \delta_1^*.$$

**Theorem.** *Let  $G$  be a self-small right  $R$ -module. Then the additive functor*

$$h_G(\cdot) : G\text{-Plex} \rightarrow \mathbf{Mod}\text{-End}_R(G)$$

*is a category equivalence.*

Thus the category of right  $\text{End}_R(G)$ -modules,  $\mathbf{Mod}\text{-End}_R(G)$ , is characterized in terms of a category  $G\text{-Plex}$  in which  $G$  is a small projective generator.

One of the more attractive elements of this point of view is that it dualizes without too much effort. We will assume the set theoretic condition

$(\mu)$  measurable cardinals do not exist.

The assumption  $(\mu)$  is true under Gödel’s constructibility hypothesis. Under  $(\mu)$  we can make a complete dualization of the above theorem. The right  $R$ -module  $G$  is *self-slender* if for each index set  $\mathcal{I}$  and  $R$ -module map  $\phi : G^{\mathcal{I}} \rightarrow G$  there is a finite set  $\mathcal{J} \subset \mathcal{I}$  such that

$$G^{\mathcal{I} \setminus \mathcal{J}} \subset \ker \phi.$$

Equivalently  $G$  is self-slender if for each index set  $\mathcal{I}$  the canonical map

$$\text{Hom}_R(G, G)^{(\mathcal{I})} \rightarrow \text{Hom}_R(G^{\mathcal{I}}, G)$$

is an isomorphism. Let

$$\mathcal{W} = W_0 \xrightarrow{\sigma_1} W_1 \xrightarrow{\sigma_2} W_2 \xrightarrow{\sigma_3} \dots$$

be a complex of right  $R$ -modules. Then  $\mathcal{W}$  is a  $G$ -complex if  $W_k$  is a direct summand of a direct product of copies of  $G$  for each integer  $k \geq 0$ , and if it satisfies the lifting property that is dual to the lifting property satisfied by a  $G$ -plex. Define the category of  $G$ -complexes,  $G\text{-Copl}x$ , to be that category whose objects are  $G$ -complexes and whose maps are homotopy equivalence classes  $[f]$  of chain maps  $f$  between  $G$ -complexes. The functor

$$h^G(\cdot) : G\text{-Copl}x \rightarrow \text{End}_R(G)\text{-Mod}$$

is defined by

$$h^G(\mathcal{W}) = \text{coker } \text{Hom}_R(\partial_1, G)$$

which is just the zeroth homology group of the complex of left  $\text{End}_R(G)$ -modules  $\text{Hom}_R(\mathcal{W}, G)$ .

**Theorem.** Assume  $(\mu)$  and let  $G$  be a self-slender right  $R$ -module. Then the additive functor

$$h^G(\cdot) : G\text{-Copl}x \rightarrow \text{End}_R(G)\text{-Mod}$$

is a category equivalence.

Consequently, we have characterized the category  $\text{End}_R(G)\text{-Mod}$  of left  $\text{End}_R(G)$ -modules in terms of the category  $G\text{-Copl}x$  in which  $G$  is a slender injective cogenerator. It is worth noting that if  $G$  is a reduced torsion-free finite rank

abelian group then  $G$  is both self-small and self-slender. Thus for these groups we have a complete characterization of the right  $\text{End}_{\mathbb{Z}}(G)$ -modules and the left  $\text{End}_{\mathbb{Z}}(G)$ -modules by categories completely determined by  $G$ .

From these theorems we sample the existing module theoretic properties for  $G$  that can be characterized in terms of  $\text{End}_R(G)$ , and we look at those properties of  $\text{End}_R(G)$  that can be characterized in terms of  $G$ . Specifically we characterize the homological dimensions of  $G$  as a left  $\text{End}_R(G)$ -module, we characterize the global dimensions of  $\text{End}_R(G)$  in terms of  $G$ , and we consider ring theoretic properties for  $\text{End}_R(G)$ . E.g. we determine when  $\text{End}_R(G)$  is left or right Noetherian, left or right coherent, right or left self-injective, a left or right cogenerator ring, a left or right PF ring, a QF ring, or a left or right FP-injective ring. We also characterize those  $C$  such that  $\text{Hom}_R(G, C)$  is a projective or an injective right  $\text{End}_R(G)$ -module.

There are a few diagrams that illustrate a connection between  $G$ ,  $\text{End}_R(G)$ , homology, and point set topological spaces. This equivalence of ideas from different areas of mathematics is rare. Given a (not necessarily self-small) right  $R$ -module  $G$  there is a commutative diagram of categories and functors in which **M-spaces** denotes a category whose objects are point set topological spaces with specified homology groups. It is usual to call a space concentrated at some integer  $k \geq 0$  a *Moore  $k$ -space*. Notice that the diagram 16.1 contains the left modules and the right modules over  $\text{End}_R(G)$ , as well as the functors  $\text{Tor}^*$  and  $\text{Ext}^*$ .

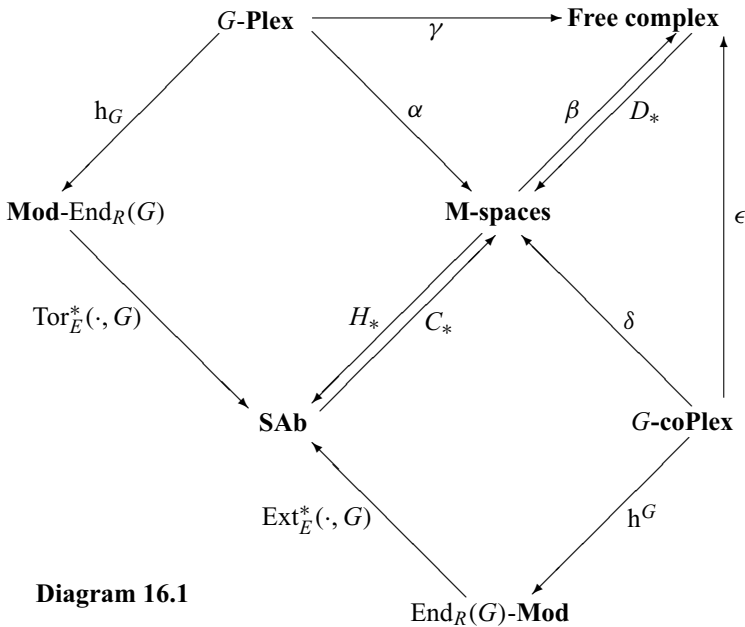


Diagram 16.1

Fix  $G$ . By letting  $X$  denote the topological space whose fundamental group is  $G$ , called an *Eilenberg–MacLane space*, we develop a commutative triangle (see Diagram 17.1) of categories and functors. When  $G$  is self-small this triangle consists of category equivalences between homology theory, modules over a ring, and a category of point set topology.



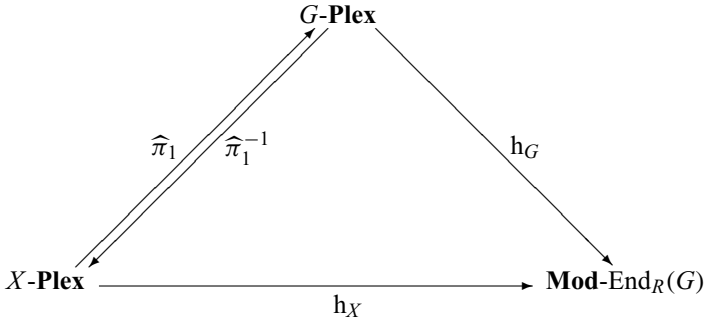


Diagram 17.1

The text ends with a chapter that combines noncommutative localization of rings with an additive functor  $\mathbf{QH}_G(\cdot)$ , and a new measurement of modules called *margimorphism* to give several right  $R$ -modules  $G$  possessing unique decompositions. For example, let  $Q_G$  denote the semi-primary classical right ring of quotients of the ring  $\text{End}_R(G)$ . Then  $G$  is margimorphic to  $G'$  iff  $\mathbf{QH}_G(G) \cong \mathbf{QH}_G(G')$  as right  $Q_G$ -modules. Furthermore, we prove that  $Q_G$  can be used to find a unique direct sum decomposition for  $G$ .

**Theorem.** *Suppose that  $\text{End}_R(G)$  possesses a semi-primary classical right ring of quotients  $Q_G$  such that  $Q_G/\mathcal{J}(Q_G)$  is a product of division rings. Then  $G$  has a unique direct sum decomposition in the sense of the Azumaya–Krull–Schmidt theorem.*

**Organization:** Aside from a preliminary chapter, the book is in three parts:

1. A portion of the book is devoted to the study of a number-theoretic connection between  $G$  and  $E(G)$ . This includes an investigation into the algebraic number theory of algebraic number fields.
2. A portion of the book is devoted to the study of the module and ideal theoretic connections between  $G$  and  $\text{End}(G)$ .
3. A portion of the book shows a categorical connection between  $G$ ,  $\text{End}(G)$ , and certain point set topological spaces.

Chapters 2–7 develop a method for utilizing the commutative property in  $E(G)$  in discussing unique direct sum decompositions of  $G$ . These techniques characterize the class number of an algebraic number field. Chapter 8 uses analytic number theory to study groups  $G$  such that each group locally isomorphic to  $G$  is isomorphic to  $G$ . Chapter 9 gives the homological framework needed to study  $\text{End}(G)$  systematically, including the theorem giving the category equivalence  $h_G : G\text{-Plex} \rightarrow \text{Mod-End}_R(G)$ . Chapter 10 gives several hypotheses under which the tensor functor  $\mathbf{T}_G$  with  $G$  is a category equivalence. In Chapter 11 we describe the ring  $\text{End}_R(G)$  with small right or left global dimension. Chapters 12–15 give characterizations of modules of the form  $\text{Hom}_R(G, C)$ . Chapters 16 and 17 give the diagrams relating abelian groups, modules,  $\text{End}_R(G)$ , and point set topology. These techniques characterize the class number of an algebraic number field. Chapter 18 is devoted to margimorphisms.

Each chapter ends with a number of exercises and the chapters themselves contain many statements of the type *the reader will prove that . . .* These are details or generalizations that I felt detracted from the discussion. The young ring or module theorist should attempt these exercises. Since examples guide our intuition and guide us to theorems, the reader should not be surprised at the number of examples used to motivate our discussions.

I would like to thank Fordham University for giving me the time and the resources needed to write this book during the years 2002–2007. I would also like to thank my colleagues for carefully reading this book in manuscript form and for their subsequent comments. Their e-mails helped me to polish soft points in the manuscript. I am especially grateful to the faculty and students at New Mexico State University at Las Cruces; to Professors D. Arnold, R. Mines, R. Hunter, E. A. Walker, C. Walker, F. Richman, C. Vinsonhaler, and W. Wickless who brought me to modern ideas for research in endomorphism rings, modules, and abelian groups; and to Professor C. Faith, who introduced me to much of the ring and module theory that the reader will find within.

Since my research style produces many  $\text{\TeX}$  files and almost no paper files, I am both author and technical typist on this project. Any errors within are my responsibility.