Cambridge University Press
978-0-521-19960-5 — Modules over Endomorphism Rings
Theodore G. Faticoni
Frontmatter
More Information

MODULES OVER ENDOMORPHISM RINGS

This is an extensive synthesis of recent work in the study of endomorphism rings and their modules, bringing together direct sum decompositions of modules, the class number of an algebraic number field, point set topological spaces, and classical noncommutative localization.

The main idea behind the book is to study modules G over a ring R via their endomorphism ring $\mathrm{End}_R(\mathrm{G})$. The author discusses a wealth of results that classify G and $\mathrm{End}_R(G)$ via numerous properties, and in particular results from point set topology are used to provide a complete characterization of the direct sum decomposition properties of G.

For graduate students this is a useful introduction, while the more experienced mathematician will discover that the book contains results that are not otherwise available. Each chapter contains a list of exercises and problems for future research, which provide a springboard for students entering modern professional mathematics.

Theodore G. Faticoni is Professor in the Mathematics Department at Fordham University, New York.

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit

http://www.cambridge.org/uk/series/sSeries.asp?code=EOM

70 A. Pietsch and J. Wenzel *Orthonormal Systems and Banach Space Geometry*
71 G. E. Andrews, R. Askey and R. Roy *Special Functions*
72 R. Ticciati *Quantum Field Theory for Mathematicians*
73 M. Stern *Semimodular Lattices*
74 I. Lasiecka and R. Triggiani *Control Theory for Partial Differential Equations I*
75 I. Lasiecka and R. Triggiani *Control Theory for Partial Differential Equations II*
76 A. A. Ivanov *Geometry of Sporadic Groups I*
77 A. Schinzel *Polynomials with Special Regard to Reducibility*
78 T. Beth, D. Jungnickel and H. Lenz *Design Theory II, 2nd edn*
79 T. W. Palmer *Banach Algebras and the General Theory of *-Algebras II*
80 O. Stormark *Lie's Structural Approach to PDE Systems*
81 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables*
82 J. P. Mayberry *The Foundations of Mathematics in the Theory of Sets*
83 C. Foias, O. Manley, R. Rosa and R. Temam *Navier–Stokes Equations and Turbulence*
84 B. Polster and G. Steinke *Geometries on Surfaces*
85 R. B. Paris and D. Kaminski *Asymptotics and Mellin–Barnes Integrals*
86 R. McEliece *The Theory of Information and Coding, 2nd edn*
87 B. A. Magurn *An Algebraic Introduction to K-Theory*
88 T. Mora *Solving Polynomial Equation Systems I*
89 K. Bichteler *Stochastic Integration with Jumps*
90 M. Lothaire *Algebraic Combinatorics on Words*
91 A. A. Ivanov and S. V. Shpectorov *Geometry of Sporadic Groups II*
92 P. McMullen and E. Schulte *Abstract Regular Polytopes*
93 G. Gierz *et al. Continuous Lattices and Domains*
94 S. R. Finch *Mathematical Constants*
95 Y. Jabri *The Mountain Pass Theorem*
96 G. Gasper and M. Rahman *Basic Hypergeometric Series, 2nd edn*
97 M. C. Pedicchio and W. Tholen (eds.) *Categorical Foundations*
98 M. E. H. Ismail *Classical and Quantum Orthogonal Polynomials in One Variable*
99 T. Mora *Solving Polynomial Equation Systems II*
100 E. Olivieri and M. Eulália Vares *Large Deviations and Metastability*
101 A. Kushner, V. Lychagin and V. Rubtsov *Contact Geometry and Nonlinear Differential Equations*
102 L. W. Beineke and R. J. Wilson (eds.) with P. J. Cameron *Topics in Algebraic Graph Theory*
103 O. Staffans *Well-Posed Linear Systems*
104 J. M. Lewis, S. Lakshmivarahan and S. K. Dhall *Dynamic Data Assimilation*
105 M. Lothaire *Applied Combinatorics on Words*
106 A. Markoe *Analytic Tomography*
107 P. A. Martin *Multiple Scattering*
108 R. A. Brualdi *Combinatorial Matrix Classes*
109 J. M. Borwein and J. D. Vanderwerff *Convex Functions*
110 M.-J. Lai and L. L. Schumaker *Spline Functions on Triangulations*
111 R. T. Curtis *Symmetric Generation of Groups*
112 H. Salzmann, T. Grundhöfer, H. Hähl and R. Löwen *The Classical Fields*
113 S. Peszat and J. Zabczyk *Stochastic Partial Differential Equations with Lévy Noise*
114 J. Beck *Combinatorial Games*
115 L. Barreira and Y. Pesin *Nonuniform Hyperbolicity*
116 D. Z. Arov and H. Dym *J-Contractive Matrix Valued Functions and Related Topics*
117 R. Glowinski, J.-L. Lions and J. He *Exact and Approximate Controllability for Distributed Parameter Systems*
118 A. A. Borovkov and K. A. Borovkov *Asymptotic Analysis of Random Walks*
119 M. Deza and M. Dutour Sikirić *Geometry of Chemical Graphs*
120 T. Nishiura *Absolute Measurable Spaces*
121 M. Prest *Purity, Spectra and Localisation*
122 S. Khrushchev *Orthogonal Polynomials and Continued Fractions: From Euler's Point of View*
123 H. Nagamochi and T. Ibaraki *Algorithmic Aspects of Graph Connectivity*
124 F. W. King *Hilbert Transforms I*
125 F. W. King *Hilbert Transforms II*
126 O. Calin and D.-C. Chang *Sub-Riemannian Geometry*
127 M. Grabisch, J.-L. Marichal, R. Mesiar and E. Pap *Aggregation Functions*
128 L. W. Beineke and R. J. Wilson (eds) with J. L. Gross and T. W. Tucker *Topics in Topological Graph Theory*
129 J. Berstel, D. Perrin and C. Reutenauer *Codes and Automata*

Modules over Endomorphism Rings

THEODORE G. FATICONI
Fordham University

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521199605

First published 2010

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-19960-5 Hardback

To my wife Barbara Jean
who helped me read our book of life.

Cambridge University Press
978-0-521-19960-5 — Modules over Endomorphism Rings
Theodore G. Faticoni
Frontmatter
More Information

Contents

Cambridge University Press
978-0-521-19960-5 — Modules over Endomorphism Rings
Theodore G. Faticoni
Frontmatter
More Information

Preface

The chapters in this book are from papers published or submitted to peer-reviewed journals. These papers were written by the author during the calendar years 2006–2008.

There is a simple example that motivates the point of view of this text. Let $\mathbf{k}$ be a field, let V be an n-dimensional $\mathbf{k}$-vector space for some integer $n > 0$, and let $E = \mathrm{Mat}_n(\mathbf{k})$ denote the ring of $n \times n$-matrices over $\mathbf{k}$. Fix an ordered basis β for V and let $[\mathbf{v}]_\beta$ denote the vector representation for $\mathbf{v}$ relative to β. Given $r \in E$ and $\mathbf{v} \in V$ then we define

$$r\mathbf{v} = r \cdot [\mathbf{v}]_\beta,$$

where $\cdot$ is the usual multiplication between the $n \times n$ matrix r and the column vector $[\mathbf{v}]_\beta$. This multiplication makes V a left E-module. Given a right ideal $I \subset E$ we define

$$IV = \left\{ \sum_i r_i \mathbf{v}_i \,\middle|\, \text{finitely many elements } r_i \in I \text{ and } \mathbf{v}_i \in V \right\}.$$

Then the assignment

$$I \longmapsto IV$$

defines a bijection between the set of right ideals of E and the set of $\mathbf{k}$-subspaces of V. Thus we can study some properties of V by studying the right ideals in the ring E. Notice that we have passed from a strictly additive setting into a setting that is additive and multiplicative. This gain in stucture improves our chances of solving certain problems concerning V.

There is little hope of generalizing the bijection $I \longmapsto IV$ to more general modules over associative rings without sacrificing something, so we hope for the best possible generalization. To find this generalization we will use elements from ring theory, module theory, and some elementary homology and homotopy theory of complexes over associative rings, and in at least a couple of instances we use some point set topology. More details follow.

Most modern research into direct sum decompositions of reduced torsion-free finite rank abelian groups (now called *rtffr groups*) begins with a study of projective modules over $\text{End}(G)$. This stems from the Arnold–Lady theorem, which shows that direct summands of G^n correspond to projective direct summands of $\text{End}(G)^n$.

This method begins a study of modules G over a ring R and the endomorphism ring $\text{End}_R(G)$. We begin by constructing a category G-plex of what are called G-plexes. This category is category equivalent to **Mod**-$\text{End}_R(G)$, which makes G-plex the category to be studied if we wish to characterize G or $\text{End}_R(G)$. A duality is used to characterize $\text{End}_R(G)$**-Mod** in terms of a category G-coplex. Thus we can characterize the rings $\text{End}_R(G)$ whose properties are on the following list of properties of rings:

1. right or left hereditary
2. right or left Noetherian
3. right or left coherent
4. right or left FP-injective
5. right or left self-injective
6. right or left cogenerator
7. right PF rings
8. QF rings

We consider $\text{End}_R(G)$ and the left $\text{End}_R(G)$-module G, and we characterize several integers associated with rings and modules. Specifically we characterize the integers on the following list:

1. projective dimension of G
2. injective dimension of G
3. flat dimension of G
4. right or left global dimension of $\text{End}_R(G)$

Several properties are left as exercises.

One of the purposes of this book is to show that we can study groups locally isomorphic to G by studying invertible fractional right ideals of $E(G)$ where

$$E(G) = \text{End}(G)/\mathcal{N}(\text{End}(G)).$$

For example, let $n > 0$ be an integer, and given a commutative prime ring R, let $\text{Pic}(R)$ denote the abelian group of isomorphism classes of invertible fractional right ideals of R. If G is a strongly indecomposable *rtffr* group and if $E(G)$ is commutative then the set of isomorphism classes (H) of groups H that are *locally isomorphic* to G^n is bijective with the finite abelian group $\text{Pic}(E(G))$. In this setting we show that if H, K, and L are direct summands of G^n, then cancellation in the isomorphism $H \oplus K \cong H \oplus L$ can be viewed as cancellation of elements in the abelian group $\text{Pic}(E(G))$.

This point of view gives a new insight into the problem of finding the class number $h(\mathbf{k})$ of the algebraic number field $\mathbf{k}$. For example, those $\mathbf{k}$ with $h(\mathbf{k}) = 1$ are classified in the following result. Let $\overline{E}$ denote the algebraic integers in $\mathbf{k}$. Let $\Omega(\overline{E}) = \{$rtffr groups $G \mid \text{End}(G) \cong \overline{E}\}$.

Theorem. *The following are equivalent for the algebraic number field* $\mathbf{k}$.

1. $h(\mathbf{k}) = 1$.
2. *Each* $G \in \Omega(\mathbf{k})$ *has the power cancellation property. (For each integer* $m > 0$ *and group* H, $G^m \cong H^m$ *implies that* $G \cong H$.)
3. *Each group* H *that is locally isomorphic to* G *is isomorphic to* G.

Some research over the thirty-year period from 1970 to 2000 dealt with the rtffr groups G that were finitely generated left $\text{End}(G)$-modules, or projective left $\text{End}(G)$-modules, or that had right hereditary endomorphism ring. Our thirty-odd pages on this type of result give us a unified approach to these problems and extends existing results. Subsequently, we use the machinery developed in Chapter 9 to characterize the left $\text{End}(G)$-module G that possesses some properties from the following list:

1. finitely generated
2. finitely presented
3. coherent
4. projective
5. quasi-projective
6. possesses a projective cover
7. cogenerator
8. generator
9. progenerator
10. quasi-progenerator
11. Noetherian

Let R be an associative ring with identity, let G be a right R-module, and let

$$\text{End}_R(G)$$

denote the ring of R-endomorphisms of G. The module G is *self-small* if for each index set $\mathcal{I}$ and each R-module map $\phi : G \longrightarrow G^{(\mathcal{I})}$ there is a finite set $\mathcal{J} \subset \mathcal{I}$ such that $\phi(G) \subset G^{(\mathcal{J})}$. In other words there is a natural isomorphism

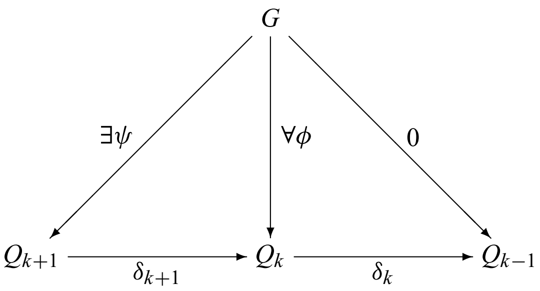
$$\text{Hom}_R(G, G)^{(\mathcal{I})} \longrightarrow \text{Hom}_R(G, G^{(\mathcal{I})}).$$

Let $\mathbf{P}(G) = \{$right R-modules $Q \mid Q \oplus Q' \cong G^{(\mathcal{I})}$ for some index set $\mathcal{I}$ and some right R-module $Q'\}$. A *G-plex* is a complex

$$\mathcal{Q} = \cdots \xrightarrow{\delta_3} Q_2 \xrightarrow{\delta_2} Q_1 \xrightarrow{\delta_1} Q_0$$

with the properties that

1. $Q_k \in \mathbf{P}(G)$ for each $k \geq 0$ and
2. G has the following lifting property for each $k \geq 1$. Given a map $\phi : G \longrightarrow Q_k$ such that $\delta_k \phi = 0$ there is a map $\psi : G \longrightarrow Q_{k+1}$ such that $\phi = \delta_{k+1}\psi$ as in the commutative triangle



of right R-modules.

The category of G-plexes G-**Plex** is the additive category whose objects are the G-plexes $\mathcal{Q}$ and whose morphisms are homotopy equivalence classes $[f]$ of chain maps

$$f : \mathcal{Q} \longrightarrow \mathcal{Q}'$$

between G-plexes $\mathcal{Q}$ and $\mathcal{Q}'$.

If we let **Mod**-$\mathrm{End}_R(G)$ denote the category of right $\mathrm{End}_R(G)$-modules then the functor

$$\mathrm{h}_G(\cdot) : G\text{-}\mathbf{Plex} \longrightarrow \mathbf{Mod}\text{-}\mathrm{End}_R(G)$$

sends $\mathcal{Q} \in G$-**Plex** to the zeroth homology group of the complex

$$\mathrm{Hom}(G, \mathcal{Q}) = \cdots \xrightarrow{\delta_2^*} \mathrm{Hom}_R(G, Q_1) \xrightarrow{\delta_1^*} \mathrm{Hom}_R(G, Q_0),$$

or in other words

$$\mathrm{h}_G(\mathcal{Q}) = \mathrm{coker}\ \delta_1^*.$$

Theorem. *Let G be a self-small right R-module. Then the additive functor*

$$\mathrm{h}_G(\cdot) : G\text{-}\mathbf{Plex} \longrightarrow \mathbf{Mod}\text{-}\mathrm{End}_R(G)$$

is a category equivalence.

Thus the category of right $\mathrm{End}_R(G)$-modules, **Mod**-$\mathrm{End}_R(G)$, is characterized in terms of a category G-**Plex** in which G is a small projective generator.

One of the more attractive elements of this point of view is that it dualizes without too much effort. We will assume the set theoretic condition

$$(\mu) \text{ measurable cardinals do not exist.}$$

The assumption (μ) is true under Gödel's constructibility hypothesis. Under (μ) we can make a complete dualization of the above theorem. The right R-module G is *self-slender* if for each index set $\mathcal{I}$ and R-module map $\phi : G^{\mathcal{I}} \longrightarrow G$ there is a finite set $\mathcal{J} \subset \mathcal{I}$ such that

$$G^{\mathcal{I}\setminus\mathcal{J}} \subset \ker \phi.$$

Equivalently G is self-slender if for each index set $\mathcal{I}$ the canonical map

$$\mathrm{Hom}_R(G,G)^{(\mathcal{I})} \longrightarrow \mathrm{Hom}_R(G^{\mathcal{I}},G)$$

is an isomorphism. Let

$$\mathcal{W} = W_0 \xrightarrow{\sigma_1} W_1 \xrightarrow{\sigma_2} W_2 \xrightarrow{\sigma_3} \cdots$$

be a complex of right R-modules. Then $\mathcal{W}$ is a *G-coplex* if W_k is a direct summand of a direct product of copies of G for each integer $k \geq 0$, and if it satisfies the lifting property that is dual to the lifting property satisfied by a G-plex. Define *the category of G-coplexes*, G-**Coplx**, to be that category whose objects are G-coplexes and whose maps are homotopy equivalence classes $[f]$ of chain maps f between G-coplexes. The functor

$$\mathrm{h}^G(\cdot) : G\text{-}\mathbf{Coplx} \longrightarrow \mathrm{End}_R(G)\text{-}\mathbf{Mod}$$

is defined by

$$\mathrm{h}^G(\mathcal{W}) = \mathrm{coker}\ \mathrm{Hom}_R(\partial_1, G)$$

which is just the zeroth homology group of the complex of left $\mathrm{End}_R(G)$-modules $\mathrm{Hom}_R(\mathcal{W}, G)$.

Theorem. *Assume (μ) and let G be a self-slender right R-module. Then the additive functor*

$$\mathrm{h}^G(\cdot) : G\text{-}\mathbf{Coplx} \longrightarrow \mathrm{End}_R(G)\text{-}\mathbf{Mod}$$

is a category equivalence.

Consequently, we have characterized the category $\mathrm{End}_R(G)$-**Mod** of *left* $\mathrm{End}_R(G)$-modules in terms of the category G-**Coplx** in which G is a slender injective cogenerator. It is worth noting that if G is a reduced torsion-free finite rank abelian group then G is both self-small and self-slender. Thus for these groups we have a complete characterization of the right $\mathrm{End}_{\mathbf{Z}}(G)$-modules and the left $\mathrm{End}_{\mathbf{Z}}(G)$-modules by categories completely determined by G.

From these theorems we sample the existing module theoretic properties for G that can be characterized in terms of $\mathrm{End}_R(G)$, and we look at those properties of $\mathrm{End}_R(G)$ that can be characterized in terms of G. Specifically we characterize the homological dimensions of G as a left $\mathrm{End}_R(G)$-module, we characterize the global dimensions of $\mathrm{End}_R(G)$ in terms of G, and we consider ring theoretic properties for $\mathrm{End}_R(G)$. E.g. we determine when $\mathrm{End}_R(G)$ is left or right Noetherian, left or right coherent, right or left self-injective, a left or right cogenerator ring, a left or right PF ring, a *QF* ring, or a left or right *FP*-injective ring. We also characterize those C such that $\mathrm{Hom}_R(G, C)$ is a projective or an injective right $\mathrm{End}_R(G)$-module.

There are a few diagrams that illustrate a connection between G, $\mathrm{End}_R(G)$, homology, and point set topological spaces. This equivalence of ideas from different areas of mathematics is rare. Given a (not necessarily self-small) right R-module G there is a commutative diagram of categories and functors in which **M-spaces** denotes a category whose objects are point set topological spaces with specified homology groups. It is usual to call a space concentrated at some integer $k \geq 0$ a *Moore k-space*. Notice that the diagram 16.1 contains the left modules and the right modules over $\mathrm{End}_R(G)$, as well as the functors Tor^* and Ext^*.

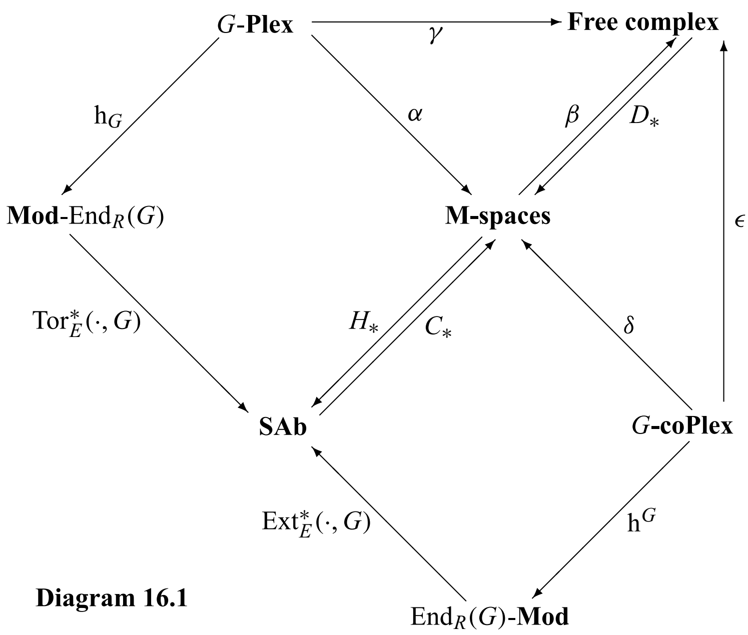


Diagram 16.1

Fix G. By letting X denote the topological space whose fundamental group is G, called an *Eilenberg–MacLane space*, we develop a commutative triangle (see Diagram 17.1) of categories and functors. When G is self-small this triangle consists of category equivalences between homology theory, modules over a ring, and a category of point set topology.

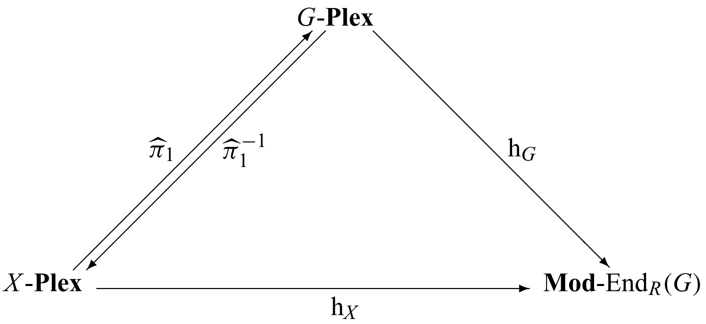


Diagram 17.1

The text ends with a chapter that combines noncommutative localization of rings with an additive functor $\mathbf{QH}_G(\cdot)$, and a new measurement of modules called *margimorphism* to give several right R-modules G possessing unique decompositions. For example, let Q_G denote the semi-primary classical right ring of quotients of the ring $\mathrm{End}_R(G)$. Then G is margimorphic to G' iff $\mathbf{QH}_G(G) \cong \mathbf{QH}_G(G')$ as right Q_G-modules. Furthermore, we prove that Q_G can be used to find a unique direct sum decomposition for G.

Theorem. *Suppose that* $\mathrm{End}_R(G)$ *possesses a semi-primary classical right ring of quotients* Q_G *such that* $Q_G/\mathcal{J}(Q_G)$ *is a product of division rings. Then G has a unique direct sum decomposition in the sense of the Azumaya–Krull–Schmidt theorem.*

Organization: Aside from a preliminary chapter, the book is in three parts:

1. A portion of the book is devoted to the study of a number-theoretic connection between G and $E(G)$. This includes an investigation into the algebraic number theory of algebraic number fields.
2. A portion of the book is devoted to the study of the module and ideal theoretic connections between G and $\mathrm{End}(G)$.
3. A portion of the book shows a categorical connection between G, $\mathrm{End}(G)$, and certain point set topological spaces.

Chapters 2–7 develop a method for utilizing the commutative property in $E(G)$ in discussing unique direct sum decompositions of G. These techniques characterize the class number of an algebraic number field. Chapter 8 uses analytic number theory to study groups G such that each group locally isomorphic to G is isomorphic to G. Chapter 9 gives the homological framework needed to study $\mathrm{End}(G)$ systematically, including the theorem giving the category equivalence $\mathrm{h}_G : G$-**Plex** $\longrightarrow$ **Mod**-$\mathrm{End}_R(G)$. Chapter 10 gives several hypotheses under which the tensor functor $\mathbf{T}_G$ with G is a category equivalence. In Chapter 11 we describe the ring $\mathrm{End}_R(G)$ with small right or left global dimension. Chapters 12–15 give characterizations of modules of the form $\mathrm{Hom}_R(G, C)$. Chapters 16 and 17 give the diagrams relating abelian groups, modules, $\mathrm{End}_R(G)$, and point set topology. These techniques characterize the class number of an algebraic number field. Chapter 18 is devoted to margimorphisms.

Each chapter ends with a number of exercises and the chapters themselves contain many statements of the type *the reader will prove that* . . . These are details or generalizations that I felt detracted from the discussion. The young ring or module theorist should attempt these exercises. Since examples guide our intuition and guide us to theorems, the reader should not be surprised at the number of examples used to motivate our discussions.

I would like to thank Fordham University for giving me the time and the resources needed to write this book during the years 2002–2007. I would also like to thank my colleagues for carefully reading this book in manuscript form and for their subsequent comments. Their e-mails helped me to polish soft points in the manuscript. I am especially grateful to the faculty and students at New Mexico State University at Las Cruces; to Professors D. Arnold, R. Mines, R. Hunter, E. A. Walker, C. Walker, F. Richman, C. Vinsonhaler, and W. Wickless who brought me to modern ideas for research in endomorphism rings, modules, and abelian groups; and to Professor C. Faith, who introduced me to much of the ring and module theory that the reader will find within.

Since my research style produces many TEX files and almost no paper files, I am both author and technical typist on this project. Any errors within are my responsibility.