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Excerpt
More Information

1
Preliminary results

This text assumes that the reader is familiar with abelian groups and unital modules over associative rings with unity as contained in the texts by L. Fuchs [59] and F. W. Anderson and K. R. Fuller [7]. We will reference but not prove those results that we feel fall outside of the line of thought of this book. I suggest that you use this chapter as a reference and nothing more. Skim through this chapter. Do not attempt to plow through these results as though they were exercises.

1.1 Rings, modules, and functors

We will deal with several rings at once in our discussions so we will use more than one symbol to denote rings. Thus R and E denote rings. Given right R-modules G and H and an index set $\mathcal{I}$, let $c = \text{card}(\mathcal{I})$. For each $i \in \mathcal{I}$ let $G_i \cong G$. Then

$$G^{(c)} = G^{(\mathcal{I})} = \oplus_{i\in\mathcal{I}} G_i$$
$$G^c = G^{\mathcal{I}} = \prod_{i\in\mathcal{I}} G_i$$

are the usual *direct sum* and *direct product* of c copies of G. We say that G is *indecomposable* if $G \cong H \oplus K$ implies that $H = 0$ or $K = 0$. The abelian group G is said to be *strongly indecomposable* if each subgroup of finite index in G is an indecomposable group.

Let G and H be right R-modules. As usual

$$\text{End}_R(G)$$

is *the ring of R-endomorphisms* $f : G \longrightarrow G$ and

$$\text{Hom}_R(G, H)$$

is the group of R-module homomorphisms $f : G \longrightarrow H$. We consider G as a left $\text{End}_R(G)$-module by setting $f \cdot x = f(x)$ for $f \in \text{End}_R(G)$ and $x \in G$. Also $\text{Hom}_R(G, H)$ is a right $\text{End}_R(G)$-module if we define $x \cdot f = x \circ f$ for each

$x \in \mathrm{Hom}_R(G, H)$ and $f \in \mathrm{End}_R(G)$. A right module over $\mathrm{End}_R(G)$ is called an *endomorphism module*. Specifically, $\mathrm{Hom}_R(G, H)$ is an endomorphism module.

The right R-module G is *self-small* if for each cardinal c the canonical injection

$$\mathrm{Hom}_R(G, G)^{(c)} \longrightarrow \mathrm{Hom}_R(G, G^{(c)})$$

is an isomorphism. Equivalently, given an index set $\mathcal{I}$ and an R-module map $\phi : G \longrightarrow G^{(\mathcal{I})}$ there is a finite subset $\mathcal{J} \subset \mathcal{I}$ such that $\phi(G) \subset G^{(\mathcal{J})}$. Finitely generated modules are self-small, as are the abelian subgroups of finite-dimensional $\mathbf{Q}$-vector spaces. (Such groups are called *torsion-free finite rank groups* in the literature). The quasi-cyclic group $\mathbf{Z}(p^\infty)$ is not self-small for primes $p \in \mathbf{Z}$. (See [59].)

Let E be ring. *The Jacobson radical of E* is the ideal $\mathcal{J}(E)$ defined as follows.

$$\begin{aligned} \mathcal{J}(E) &= \cap\{M \mid M \subset E \text{ is a maximal right ideal }\} \\ &= \cap\{M \mid M \subset E \text{ is a maximal left ideal }\} \\ &= \{r \in E \mid 1 + rx \text{ is a unit in } E \text{ for each } x \in E\} \end{aligned}$$

In particular, if $J \subset E$ is a right ideal such that $J + \mathcal{J}(E) = E$ then $J = E$.

We say that E is a *local ring* if any of the following equivalent properties hold.

1. E possesses a unique maximal right ideal M.
2. $\mathcal{J}(E)$ is the unique maximal right ideal of E.
3. $u \in E$ is a unit of E iff $u \notin \mathcal{J}(E)$.

Nakayama's theorem 1.1. *Let M be a finitely generated right R-module and let $N \subset M$ be an R-submodule of M. If $N + M\mathcal{J}(R) = M$ then $M = N$.*

The right ideal $I \subset E$ is a *nil right ideal* if each $x \in I$ is *nilpotent*. That is, for each $x \in I$ there exists an integer n such that $x^n = 0$. The *nilradical of E* is the ideal $\mathcal{N}(E)$ that is defined as follows.

$$\begin{aligned} \mathcal{N}(E) &= \{x \in E \mid xE \text{ is a nilpotent right ideal in } E\} \\ &= \{x \in E \mid Ex \text{ is a nilpotent left ideal in } E\} \end{aligned}$$

Let $x \in E$. Since $x^n = 0$ implies that $1 - x$ is a unit in E (show that one, reader), $1 - xy$ is a unit of E for each $x \in \mathcal{N}(E)$ and $y \in E$. Thus

$$\mathcal{N}(E) \subset \mathcal{J}(E).$$

The right R-module P is *projective* if for each surjection $K \xrightarrow{f} L \to 0$ of right R-modules, each mapping $g : M \to L$ lifts to a mapping $h : M \to K$ such that $fh = g$. The free R-module F for some cardinal c is projective, as is any direct summand of F. In fact, every projective right R-module is a direct summand of a free right R-module.

An *idempotent* is an element $e \in E$ such that $e^2 = e$. Often we will avoid the term idempotent and just write $e^2 = e$. If $e^2 = e \in E$ then eE is a cyclic projective right E-module, and every cyclic projective has this form. Given a ring E and an $e^2 = e \in E$ then

$$\boxed{eEe = \{exe \mid x \in E\}}$$

is a ring with identity

$$1_{eEe} = e.$$

Suppose $S \subset R$ are rings. Then S is a *unital subring* of R if $1_S = 1_R$. Although $eEe \subset E$, eEe is not a unital subring of E unless $e = 1$. There are some relationships between eEe and E.

Lemma 1.2. [7, Proposition 5.9] *Let E be a ring and let $e^2 = e$. Then* $\text{End}_E(eE) = eEe$.

The ring E is *semi-perfect* if

1. $E/\mathcal{J}(E)$ is semi-simple Artinian and
2. Given an $\bar{e}^2 = \bar{e} \in E/\mathcal{J}(E)$ there is an $e^2 = e \in E$ such that $\bar{e} = e + \mathcal{J}(E)$. That is, *idempotents lift modulo* $\mathcal{J}(E)$.

See [7, Chapter 7, §27] for a complete discussion of semi-perfect rings and their modules. Fields, local rings, and Artinian rings are semi-perfect rings. $\mathbf{Z}$ is not semi-perfect but the localization of $\mathbf{Z}$ at a prime p, $\mathbf{Z}_p$, is semi-perfect.

The next result follows from [7, Theorem 27.11].

Lemma 1.3. *Let E be a semi-perfect ring and let P be a projective right E-module.*

1. *P is indecomposable iff* $\text{End}_E(P)$ *is a local ring.*
2. *P is a direct sum of cyclic right E-modules with local endomorphism rings.*

1.2 Azumaya–Krull–Schmidt theorem

Let G be a right R-module. The purpose of this section is to show that under some conditions direct sum decompositions of G are well behaved, in a sense that we will make precise below.

An *indecomposable decomposition* of G is a direct sum

$$\boxed{G = G_1 \oplus \cdots \oplus G_t}$$

for some integer $t > 0$ and indecomposable right R-modules $G_1, \ldots, G_t$.

We say that G has a *unique decomposition* if

1. G has an indecomposable decomposition $G \cong G_1 \oplus \cdots \oplus G_t$, and
2. given another indecomposable decomposition $G \cong H_1 \oplus \cdots \oplus H_s$ then $s = t$ and there is a permutation π of the subscripts $\{1, \ldots, t\}$ such that $G_i \cong H_{\pi(i)}$ for each $i = 1, \ldots, t$.

In this case we call $G_1 \oplus \cdots \oplus G_t$ *the unique decomposition of* G. Notice that the unique decomposition of G is necessarily indecomposable. Professional mathematicians believe that modules possessing a unique decomposition are rare.

The Azumaya–Krull–Schmidt theorem is the most referenced result on the subject of the existence of a unique decomposition for a module. A proof can be found in [7].

The Azumaya–Krull–Schmidt theorem 1.4. [7, Theorem 12.6] *Suppose that $G = G_1 \oplus \cdots \oplus G_t$ is an indecomposable decomposition of right R-modules such that $\mathrm{End}_R(G_i)$ is a local ring for each $i = 1, \ldots, t$.*

1. *The direct sum $G = G_1 \oplus \cdots \oplus G_t$ is a unique decomposition of G,*
2. *If $G = H \oplus K$ then there is an $\mathcal{I} \subset \{1, \ldots, t\}$ such that $H \cong \oplus_{i \in \mathcal{I}} G_i$.*
3. *$G \cong H \oplus K \cong H \oplus N \Longrightarrow K \cong N$ for any right R-modules H, K, and N.*

Thus direct sum decompositions of projective modules over semi-perfect rings are unique.

1.3 The structure of rings

We present in this section a few results on the structure of a ring and its modules. The first shows that if S is a commutative ring then an S-module M is 0 iff it is locally zero. Thus a function is an epimorphism (or a monomorphism) iff it is locally an epimorphism (or a monomorphism).

Theorem 1.5. [97, Theorem 3.80] *Let S be a commutative ring and let M be a finitely generated S-module. Then $M = 0$ iff $M_I = 0$ for each maximal ideal $I \subset S$.*

Corollary 1.6. [97, exercise 9.22] *Let S be a commutative ring and let M be a finitely presented S-module. Then M is projective (or a generator) iff M_I is projective (or a generator) for each maximal ideal $I \subset S$.*

The next two results give a direct sum decomposition of a finite-dimensional S-algebra, where S is a commutative ring.

Wedderburn's theorem 1.7. [9, Theorem 14.1] *Let A be an Artinian $\mathbf{k}$-algebra over some field $\mathbf{k}$. Then there is a semi-simple $\mathbf{k}$-subalgebra B of A such that $A = B \oplus \mathcal{N}(A)$ as $\mathbf{k}$-vector spaces.*

Theorem 1.8. [9, Beaumont–Pierce theorem, Theorem 14.2] *Let E be a rtffr ring. Then there is a semi-prime subring T of E such that $T \oplus \mathcal{N}(E)$ has finite index in E as groups.*

We will need to count the number of isomorphism classes of right ideals in a ring R. To do this we use a result due to Jordan and Zassenhaus.

Lemma 1.9. [94, Jordan–Zassenhaus lemma, Lemma 26.3] *Let E be a semi-prime rtffr ring. Then there are at most finitely many isomorphism classes of right ideals of E.*

It will be necessary to use the fact that each rtffr ring E is $\text{End}(G)$ for some rtffr group G. The results of Butler and Corner are most often referenced in this regard.

Butler's theorem 1.10. [46, Theorem I.2.6] *If E is an rtffr ring whose additive structure is a locally free abelian group then $E \cong \text{End}(G)$ for some group $E \subset G \subset \mathbf{Q}E$.*

Theorem 1.11. [46, Theorem F.1.1] *If E is an rtffr ring then there is an rtffr group G of rank $2 \cdot \text{rank}(E)$ such that $E \cong \text{End}(G)$.*

Theorem 1.12. [56] *If M is a countable reduced torsion-free left E-module and if $E = \{q \in \mathbf{Q}E \mid qM \subset M\}$ then there is a short exact sequence*

$$0 \longrightarrow M \longrightarrow G \longrightarrow \mathbf{Q}E \oplus \mathbf{Q}E \longrightarrow 0$$

of left E-modules such that $E \cong \text{End}(G)$.

1.4 The Arnold–Lady theorem

Let G be a right R-module, let $\mathbf{Mod}\text{-}R$ denote the category of right R-modules, let

$$\boxed{\mathbf{Mod}\text{-}\text{End}_R(G) = \text{the category of right } \text{End}_R(G)\text{-modules}}$$

and let

$$\boxed{\text{End}_R(G)\text{-}\mathbf{Mod} = \text{the category of left } \text{End}_R(G)\text{-modules.}}$$

While characterizing module theoretic properties of G in terms of $\text{End}_R(G)$ will not be easy, the theorem of Arnold–Lady shows us that we can characterize direct summands of G as projective $\text{End}_R(G)$-modules. With this tool we can characterize properties surrounding the direct sum decompositions of G in terms of direct sum decompositions of projective right $\text{End}_R(G)$-modules. Finitely generated projective modules, at least on the surface, seem to be easier to work with than more general modules.

Let

$$\boxed{\begin{array}{c}\mathbf{P}(G) = \{H \mid G^{(c)} \cong H \oplus K \text{ for some cardinal } c > 0 \\ \text{and some right } R\text{-module } K\}.\end{array}}$$

We consider $\mathbf{P}(G)$ to be a full subcategory of the category $\mathbf{Mod}$-R of right R-modules. Similarly, given a ring E,

$$\boxed{\mathbf{P}(E) = \text{category of projective right } E\text{-modules.}}$$

Define additive functors

$$\mathbf{T}_G(\cdot) = \cdot \otimes_{\mathrm{End}_R(G)} G \qquad \mathbf{H}_G(\cdot) = \mathrm{Hom}_R(G, \cdot)$$
$$\mathbf{H}_G(\cdot) : \mathbf{Mod}\text{-}R \longrightarrow \mathbf{Mod}\text{-}\mathrm{End}_R(G)$$
$$\mathbf{T}_G(\cdot) : \mathbf{Mod}\text{-}\mathrm{End}_R(G) \longrightarrow \mathbf{Mod}\text{-}R.$$

That is, $\mathbf{H}_G(\cdot)$ takes right R-modules to right $\mathrm{End}_R(G)$-modules, and $\mathbf{T}_G(\cdot)$ takes right $\mathrm{End}_R(G)$-modules to right R-modules. Associated with $\mathbf{H}_G(\cdot)$ and $\mathbf{T}_G(\cdot)$ are the natural transformations

$$\Theta : \mathbf{T}_G\mathbf{H}_G(\cdot) \longrightarrow 1$$
$$\Psi : 1 \longrightarrow \mathbf{H}_G\mathbf{T}_G(\cdot)$$

defined by

$$\Theta_H(f \otimes x) = f(x)$$
$$\Psi_M(x)(\cdot) = \cdot \otimes x$$

for each $f \in \mathrm{Hom}_R(G, M)$ and $x \in G$.

$$\mathbf{T}_G \circ \mathbf{H}_G(\cdot) = \mathrm{Hom}_R(G, \cdot) \otimes_{\mathrm{End}_R(G)} G.$$
$$\mathbf{H}_G \circ \mathbf{T}_G(\cdot) = \mathrm{Hom}_R(G, \cdot_{\mathrm{End}_R(G)} G).$$

A good exercise is to demonstrate that

$$\boxed{\Theta_{\mathbf{T}_G(M)} \circ \mathbf{T}_G(\Psi_M) = 1_{\mathbf{T}_G(M)}}$$

for each right $\mathrm{End}_R(G)$-module M.

One of the themes in this text is to identify categories of modules $\mathcal{C}$ and $\mathcal{D}$ such that Θ_H and Ψ_M are isomorphisms for each $H \in \mathcal{C}$ and $M \in \mathcal{D}$. The first such result is the theorem of Arnold–Lady.

The Arnold–Lady theorem 1.13. [9, Theorem 7.21] *If G is a self-small right R-module then the functors*

$$\mathbf{H}_G(\cdot) : \mathbf{P}(G) \longrightarrow \mathbf{P}(\mathrm{End}_R(G))$$
$$\mathbf{T}_G(\cdot) : \mathbf{P}(\mathrm{End}_R(G)) \longrightarrow \mathbf{P}(G)$$

are inverse category equivalences.

Proof: For the sake of the argument let $E = \mathrm{End}_R(G)$. Since Θ and Ψ are natural transformations $\Theta_{H\oplus K} = \Theta_H \oplus \Theta_K$ for right E-modules H and K, and $\Psi_{M\oplus N} = \Psi_M \oplus \Psi_N$ for right $\mathrm{End}_R(G)$-modules M and N. Moreover, since G is self-small $\Theta_{G^{(I)}} = \oplus_I \Theta_G$. Thus given $H \oplus K \cong G^{(I)}$ we can prove that Θ_H is an isomorphism if we can prove that Θ_G is an isomorphism. Similarly, to show that Ψ_P is an isomorphism for each projective right E-module P, it suffices to show that Ψ_E is an isomorphism.

Consider the map

$$\Psi_E : E \longrightarrow \mathbf{H}_G\mathbf{T}_G(E).$$

Notice that $\mathbf{H}_G\mathbf{T}_G(E) \cong E$ with generator the map $f : G \longrightarrow \mathbf{T}_G(E)$ such that $f(x) = 1 \otimes x$ for each $x \in G$. Then $\Psi_E(1) = f$, which implies that Ψ_E is an isomorphism.

Recall that

$$\Theta_{\mathbf{T}_G(E)} \circ \mathbf{T}_G(\Psi_E) = 1_{\mathbf{T}_G(E)}.$$

Since Ψ_E, and so $\mathbf{T}_G(\Psi_E)$ are isomorphisms, it follows that $\Theta_{\mathbf{T}_G(E)} = \Theta_G$ is an isomorphism. Given our reductions the proof is complete.

Let

$$\boxed{\mathbf{P}_o(G) = \{H \mid G^{(n)} \cong H \oplus H' \text{ for some integer } n > 0 \text{ and some right } E\text{-module } H'\}.}$$

Similarly, given a ring E,

$$\boxed{\mathbf{P}_o(E) = \text{the category of finitely generated projective right } E\text{-modules.}}$$

Notice the missing self-small hypothesis in the next result.

Theorem 1.14. [46, Arnold–Lady Theorem] *Let G be a right E-module. The functors*

$$\mathbf{H}_G(\cdot) : \mathbf{P}_o(G) \longrightarrow \mathbf{P}_o(\mathrm{End}_R(G))$$
$$\mathbf{T}_G(\cdot) : \mathbf{P}_o(\mathrm{End}_R(G)) \longrightarrow \mathbf{P}_o(G)$$

are inverse category equivalences.

Example 1.15. Let $G = \oplus_p \mathbf{Z}_p$ where p ranges over the primes in $\mathbf{Z}$. Then $\text{End}(G) = \prod_p \mathbf{Z}_p$ is a semi-hereditary ring and G is a projective (= flat) left $\text{End}(G)$-module. Let $I = \oplus_p \mathbf{Z}_p$ be the ideal in $\text{End}(G)$. Then I is a projective ideal in $\text{End}(G)$ that is not finitely generated and such that $\mathbf{T}_G(I) = IG = G$. Inasmuchas $I \neq \text{End}(G)$ we have shown that $\mathbf{T}_G(\cdot) : \mathbf{P}(\text{End}(G)) \longrightarrow \mathbf{P}(G)$ is not a category equivalence if the self-small hypothesis is deleted from Theorem 1.13.

Example 1.16. Let $G = \mathbf{Q}^{(\aleph_o)}$ and let $E = \text{End}(G)$. Then G is a cyclic projective left E-module so $\mathbf{T}_G(I) \cong IG$ for each right ideal $I \subset \text{End}(G)$. If we let $I = \{f \in A \big| f(G)$ has finite dimension$\}$ then $I \ncong \text{End}(G)$ while

$$\mathbf{T}_G(I) = IG = G = \mathbf{T}_G(\text{End}(G))$$

but $I \neq \text{End}(G)$. This is another example of the necessity of the self-small hypothesis in Theorem 1.13 even though G has a rather restricted left $\text{End}(G)$-module structure.

2

Class number of an abelian group

The study of direct sum decompositions of abelian groups is as old as the study of abelian groups. In this chapter we study the direct sum decompositions of reduced torsion-free finite rank abelian groups, and we show that the associated direct sum problems are equivalent to a pair of deep problems in algebraic number theory.

2.1 Preliminaries

Let G be a reduced torsion-free finite-rank group. Following [46] we write *rtffr* to abbreviate the string of hypotheses *reduced torsion-free finite rank.*

At all times in this text

$$\boxed{E(G) = \mathrm{End}(G)/\mathcal{N}(\mathrm{End}(G)).}$$

We say that G is *cocommutative* if $E(G)$ is a commutative ring. If G is a cocommutative strongly indecomposable rtffr group then $E(G)$ is an rtffr Noetherian commutative integral domain. For instance, it follows from a theorem of J. D. Reid's that strongly indecomposable rank two abelian groups are cocommutative. Any group whose quasi-endomorphism ring is the Hamiltonian quaternions is not a cocommutative group. One of the consequences of the work in this chapter is that cocommutative groups occur naturally and often.

As in [46] we write *locally isomorphic* instead of *nearly isomorphic*, [9], or *in the same genus class*, [94]. Thus, groups G and H are *locally isomorphic* if for each integer $n > 0$ there are maps $f : G \longrightarrow H$ and $g : H \longrightarrow G$ and an integer m such that m is relatively prime to n and $fg = m1_H$ and $gf = m1_G$. Lattices M and N over a semi-prime rtffr ring E are *locally isomorphic* if for each integer $n > 0$ there are E-module maps $f : M \longrightarrow N$ and $g : N \longrightarrow M$ and an integer m such that m is relatively prime to n and $fg = m1_N$ and $gf = m1_M$. The *class number* of X, $h(X)$, where X is either an rtffr group or a lattice, is the number of isomorphism classes of those Y that are locally isomorphic to X.

An important result due to R. B. Warfield, Jr. is the following.

Warfield's theorem 2.1. [9, Theorem 13.9] *Let M and N be rtffr modules over an rtffr ring E. Then M is locally isomorphic to N iff $M^n \cong N^n$ for some integer $n > 0$.*

Let $\mathbf{P}_o(G) = \{\text{groups } H \mid H \oplus H' = G^m \text{ for some group } H' \text{ and some integer } m > 0\}$. We say that G satisfies the *power cancellation property* if $G^n \cong H^n$ for some group H and integer $n > 0$ implies that $G \cong H$. We say that G has *a Σ-unique decomposition* if G^n has a unique direct sum decomposition for each integer $n > 0$. Furthermore, G has *internal cancellation* if given $H, K, L \in \mathbf{P}_o(G)$ such that $H \oplus K \cong H \oplus L$ then $K \cong L$.

Let E be an *rtffr ring*, (i.e., a ring whose additive structure $(E, +)$ is an rtffr group). The semi-prime rtffr ring $\overline{E}$ is *integrally closed* if given a ring $\overline{E} \subset E' \subset \mathbf{Q}\overline{E}$ such that $E'/\overline{E}$ is finite then $\overline{E} = E'$.

Let (X) be the isomorphism class of X, and let

$$\Gamma(X) = \{(Y) \mid Y \text{ is locally isomorphic to } X\}.$$

Let $\mathbf{P}_o(E)$ be the set of finitely generated projective right E-modules. The local isomorphism class of X is denoted by $[X]$.

Lemma 2.2. *Let $P, Q \in \mathbf{P}_o(R)$, and suppose that $J \subset \mathcal{J}(R)$. Then $P/PJ \cong Q/QJ$ iff $P \cong Q$.*

Proof: Since P and Q are projective right R-modules, the isomorphism $P/PJ \cong Q/QJ$ lifts to a map $\phi : P \longrightarrow Q$ such that

$$\ker \phi \subset PJ \quad \text{and} \quad Q = \phi(P) + QJ.$$

By Nakayama's theorem 1.1, $Q = \phi(P)$. Since Q is a projective R-module, $P = U \oplus \ker \phi$ where $U \cong Q$. Furthermore, since $\ker \phi \subset PJ$, Nakayama's theorem shows us that $\ker \phi = 0$, whence $P \cong Q$. The converse is clear so the proof is complete.

Theorem 2.3. [19, Proposition 2.12] *Let R be an rtffr ring. The functor*

$$A_R(\cdot) : \mathbf{P}_o(R) \longrightarrow \mathbf{P}_o(R/\mathcal{N}(R))$$

defined by

$$A_R(\cdot) = \cdot \otimes_R R/\mathcal{N}(R)$$

is full. Furthermore, $A_R(\cdot)$ induces bijections of sets

1. $\{(P) \mid P \in \mathbf{P}_o(R)\} \longrightarrow \{(W) \mid W \in \mathbf{P}_o(R/\mathcal{N}(R))\}$*, and*
2. $\alpha_R : \{[P] \mid P \in \mathbf{P}_o(R)\} \longrightarrow \{[W] \mid W \in \mathbf{P}_o(R/\mathcal{N}(R))\}$.

Proof: Part 1 is true by [19, Proposition 2.12].

2. It is easily verified that α_R is well defined.

Say $W \in \mathbf{P}_o(R/\mathcal{N}(R))$. By part 1 there is a $P \in \mathbf{P}_o(R)$ such that $A_R(P) \cong W$. Thus α_G is a surjection. Say $\alpha_R[P] = \alpha_R[Q]$. Then $[P/P\mathcal{N}(R)] = [Q/Q\mathcal{N}(R)]$. By Warfield's theorem 2.1, $(P/P\mathcal{N}(R))^n \cong (Q/Q\mathcal{N}(R))^n$ for some integer $n > 0$.