Cambridge University Press
978-0-521-19921-6 — A Guide to Feedback Theory
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Excerpt
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Linear Systems: What You Missed the First Time

For an engineer, math is a language of unusual expressive power and concision. The first time that you studied differential equations, however, chances are high that this escaped you. That is natural. The purpose of this chapter is to help you to get in touch with the meaning behind the math of linear, time-invariant (LTI) systems.

1.1 Differential Equations Are a Natural Way to Express Time Evolution

Feedback systems are dynamic: ultimately, we are interested in the evolution of their state over time. Frequency-domain tools like the Laplace transform are wonderful aids for analysis, but before revisiting those let's examine the basic differential equation. We will see that despite appearances, it is quite a natural way to describe the time evolution of a dynamic system.

1.1.1 A First-Order System

Consider the RC circuit shown in Figure 1.1. The situation is that the capacitor has a charge Q_0 while the switch is open. The switch is then closed, which connects a resistor of value R across the terminals of the capacitor. To determine what happens next, we have at least two approaches. First, we can argue on physical grounds that eventually the capacitor must completely discharge, leaving the zero voltage across the capacitor. A sophisticated observer might even point out that the capacitor will never *fully* discharge, or alternatively, that a complete discharge would take an infinite amount of time. A second approach is to not bother at all with "intuitive" reasoning and

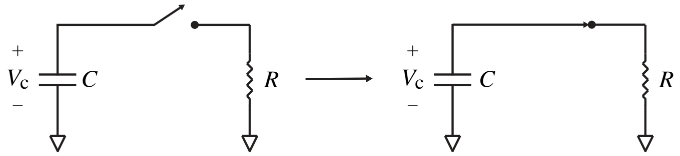

Figure 1.1 Discharging a capacitor through a resistor. The capacitor has an initial charge $Q_0 = CV_0$ and begins to discharge when the switch is closed.

physical insight. We just write down the differential equation governing the system and derive an expression for $Q(t)$ in exquisite detail.

The problem with the first approach alone is that it does not always yield the level of detail that we might require. While we can say that it will take longer to discharge if the resistor R is bigger, we are helpless to say exactly how long it will take the capacitor to lose 90 percent of its charge, for example. The problem with the second approach alone is that without physical insight, the student can never progress beyond solving little, well-packaged problems with neat answers. *If the engineer is ever to unleash their creativity to invent, design, build, and discover new things, they will be powerfully aided by understanding the yin-and-yang interplay between physical insight and mathematical analysis.*

The simple discharging of a capacitor is a great way to start understanding this balance. Starting with the initial conditions, we have an open switch and a capacitor with charge Q_0. The physical meaning of capacitance is it tells us how much charge we must supply if we are to establish a potential difference between two conductors. The greater the capacitance, the more charge we must supply to establish a given potential difference. This is beautifully and succinctly captured by the constitutive law for capacitors, $Q = CV$. We know therefore that before we throw the switch, the voltage across the capacitor terminals is $V_0 = Q_0/C$.

When we do throw the switch, we have a new constitutive relation to satisfy, namely, Ohm's law. In the first instant after the switch is closed, the charge flows through the resistor at a rate of $I = V_0/R = Q_0/RC$ Coulombs per second. But as soon as the first tiny bit of charge is removed from the capacitor, the voltage across the capacitor goes down, causing the current to decrease, which nevertheless continues to remove charge from the capacitor, and so on and so forth. At this point we have a good physical understanding of what is happening. How can we ever find out *exactly* how the charge decays with time? The approach is to express our physical insight mathematically. Almost immediately, though, we run head-first into the problem of how to deal with the progression of time. Things would be easier if time moved forward in discrete chunks. We cannot help, for example, talking about the current flow "in the first instant" after the switch is closed. Unfortunately, we know (or, at least, have no reason to doubt) that time moves forward in a continuous progression. Before we've written our first equation, then, we have a seemingly good reason to despair.

The key insight is to realize that for any "continuous" variable, there is a level of granularity beyond which a discretized representation is, for all *practical* purposes, indistinguishable from a continuous one. We know that water, for example, is composed of discrete water molecules, yet to our unaided senses the granularity is so fine as to be indistinguishable from a continuous liquid. The time variable is no different. Suppose that nature actually moved in steps of one femtosecond (10^{-15} seconds). Would we be any the wiser, even if using the fastest oscilloscopes available at the time of this writing?[1] There might be other ways of telling if nature is secretly discretizing time, but to an engineer with an oscilloscope, there is no practical difference between a universe that discretizes time in one-femtosecond chunks and one that moves forward continuously in time. This critical realization helps us to move forward, and ultimately leads us to the shorthand that we now know as differential equations.

Returning to our problem, we might consider breaking time up into tiny chunks of duration Δt. If we know the charge on the capacitor at time t, we ask "What is the charge at time $t + \Delta t$?" If we know the answer to this question in general, and we know the answer at time $t = 0$ (or some other initial time), then we know the answer for all time. So we write

$$Q_c(t + \Delta t) = Q_c(t) - I(t)\Delta t. \tag{1.1}$$

That is, the charge at the next instant is equal to the charge at the current instant, minus the charge that was bled off in one interval of time due to the current at time t. What is value of Δt? At this point we don't bother about it. We keep firmly in our mind that it is small enough so as to be indistinguishable from continuous time, and don't go back and pick a value of Δt until we need numerical answers. And what about the fact that in the truly continuous system, I does not stay constant over any interval of time? It is true that this will introduce an error. What is important is that we can make this error arbitrarily small by making Δt as small as we like. Remember, the goal is *not* to come up

[1] The Agilent DSO91304A Infiniium oscilloscope samples at a "pedestrian" 40 GSamples/second, or once every 25,000 femtoseconds.

with a model that is as accurate as nature is. That is impossible. We need only be as accurate as we can conceivably measure.

Now, since voltage is what we actually measure, we recast Eq. 1.1 in terms of the capacitor voltage

$$CV_c(t+\Delta t) = CV_c(t) - I(t)\Delta t. \tag{1.2}$$

We seize on the fact that $I(t)$ is linked to V_c through Ohm's law: $I(t) = V_c/R$. Substituting and gathering terms, we arrive at

$$\frac{V_c(t+\Delta t) - V_c(t)}{\Delta t} + \frac{1}{RC}V_c(t) = 0. \tag{1.3}$$

And now we appear to be stuck. There is no way to derive an expression for $V_c(t)$ that satisfies this equation. The best we can do is guess at a solution, plug it in, and check to see if it "works" by resulting in an equation that is self-consistent. What is a good guess?

The good news is that we needn't guess blindly. On physical grounds, we expect $V_c(t)$ to decay with time; we expect that the rate of decay will slow with time; we expect it to asymptotically approach zero. An inspired guess, drawn from an admittedly large number of possibilities, is $V_c(t) = V_0 a^{n\cdot\Delta t}$. In this solution, n is an integer index that steps us forward in time, Δt is our time increment, and a is a key parameter. If $|a^{\Delta t}| < 1$, we satisfy all of the conditions we set forth. If we have chosen correctly, the equation will determine the value of $a^{\Delta t}$ unambiguously. This in turn validates our initial guess.

Plugging into Eq. 1.3, we have

$$\frac{V_0 a^{(n+1)\cdot\Delta t} - V_0 a^{n\cdot\Delta t}}{\Delta t} + \frac{1}{RC}V_0 a^{n\cdot\Delta t} = 0. \tag{1.4}$$

A factor of $V_0 a^{n\cdot\Delta t}$ appears in all terms. Dividing both sides of the equation by $V_0 a^{n\cdot\Delta t}$ and simplifying leads to

$$a^{\Delta t} = 1 - \frac{\Delta t}{RC}. \tag{1.5}$$

This is a critical juncture in our development. In some ways, once we have $a^{\Delta t}$ we are done. Equation 1.4 is a first-order polynomial in $a^{\Delta t}$, and in Eq. 1.5 we have an equation that gives us the roots of that polynomial. We'll see polynomials like this again when we look at discrete-time systems starting in Section 1.7. For now, the *only* reason we're continuing from here is that this is not a discrete-time system, and so we must examine the implications of allowing Δt to become arbitrarily small. Continuing, Eq. 1.5 allows us to write

$$\ln a = \frac{1}{\Delta t}\ln\left(1 - \frac{\Delta t}{RC}\right). \tag{1.6}$$

Now, what do we mean when we insist that Δt is small? It is actually meaningless to insist that Δt be "small" in an abstract sense. We must instead specify its smallness in comparison to something. In this problem, suppose that we say that Δt is small compared to the quantity RC. Why does this make sense? Rewriting Eq. 1.3 slightly, we have

$$\frac{V_c(t+\Delta t)-V_c(t)}{V_c(t)} = -\frac{\Delta t}{RC}. \tag{1.7}$$

Put into words, Eq. 1.7 says that $\Delta t \ll RC$ is equivalent to saying that the fractional change in V_c during any given time step is small. This is exactly what we should hope for if we expect to better approximate a continuously evolving system by shrinking the increment Δt.

Now we employ a trick that is very common in all the disciplines of engineering and science. On conditions such as $\Delta t \ll RC$, it is natural to substitute for $f(x_0 + \Delta x)$ a *polynomial expansion*:

$$f(x_0+\Delta x) \approx a_0 + a_1 \cdot (\Delta x) + a_2 \cdot (\Delta x)^2 + a_3 \cdot (\Delta x)^3 + \cdots \tag{1.8}$$

We like polynomial expansions because they are easy to realize in computational hardware: if you can multiply and add, you can work with polynomials. This is depressingly untrue of transcendental functions like the logarithms, exponentials, and trigonometric functions that surface with such persistence in the analysis of linear systems. When you throw in the condition that $x_0 \ll \Delta x$, the good news just gets better in that you can get excellent numerical accuracy despite truncating the polynomial expansion to a finite number of terms. In fact, we often take $x_0 \ll \Delta x$ to mean that the original function can be well approximated with only *two* terms:

$$f(x_0+\Delta x) \approx a_0 + a_1 \cdot (\Delta x). \tag{1.9}$$

This happy circumstance is extremely convenient for hand analysis. You may remember this trick as "linearization."

But let's not jump ahead. Let's conservatively "guess" that for our purposes the logarithm can be adequately captured by a third-order polynomial expansion. We'll then check later to see if it introduces unacceptable numerical error. There are many techniques for fitting polynomials. For the function $\ln(1+\Delta x)$, the author chose for data points $\Delta x \in [10^{-4}, 10^{-3.5}, 10^{-3}, 10^{-2.5}, 10^{-2}]$, which are all conspicuously small compared to 1. An elementary least-squares fit[2] results in the polynomial substitution for $\ln(1+\Delta x)$

[2] For the interested reader, an excellent treatment of least-squares fits can be found in Gilbert Strang's *Introduction to Linear Algebra*, 4th ed. (Wellesley, MA: Wellesley-Cambridge Press, 2009).

$$\approx 1.2480 \times 10^{-12} + 1.0000 \cdot \Delta x - 0.5000 \cdot (\Delta x)^2 + 0.3298 \cdot (\Delta x)^3$$
$$\approx 1.0000 \cdot \Delta x - 0.5000 \cdot (\Delta x)^2 + 0.3298 \cdot (\Delta x)^3, \quad (1.10)$$

which we claim we can use with insignificant numerical error. It is instructive to do a few calculations comparing a true evaluation of $\ln(1 + \Delta x)$ with the polynomial substitute and confirming for yourself the values of Δx for which this is really okay.

Armed with this new polynomial, we return to our original problem (Eq. 1.6) and write

$$\ln a = \frac{1}{\Delta t}\left(-\frac{\Delta t}{RC} - 0.5\left(\frac{\Delta t}{RC}\right)^2 - 0.3298\left(\frac{\Delta t}{RC}\right)^3\right). \quad (1.11)$$

Now the full implications of $\Delta t \ll RC$ can be made clear. Since $\Delta t \ll RC$ is the same thing as saying $\Delta t/RC \ll 1$, we see that the terms of $\Delta t/RC$ of second order and higher in Eq. 1.11 diminish rapidly as we make Δt smaller. We can thus go even further in our approximation and neglect these terms, keeping firmly in mind that if the error this introduces bothers us, we can always make Δt smaller and smaller until the error does not bother us. Then we are left with

$$a = e^{-1/RC}, \quad (1.12)$$

which means, at long last, that the capacitor voltage evolves as

$$V_c(n \cdot \Delta t) = V_0 e^{-n \cdot \Delta t/RC}, \quad (1.13)$$

and we can finally write

$$V_c(t) = V_0 e^{-t/RC}. \quad (1.14)$$

This is the answer that we were expecting all along. What is important is how we got here. Based on physical reasoning, we came up with a discrete-time model for the system's behavior, and showed that solutions to the difference equations of this sort (see Eq. 1.4) have the form $(a^{\Delta t})^n$. We then solved for $a^{\Delta t}$, and finally explored the consequences of allowing Δt to become arbitrarily small compared to RC.

Mathematicians have an expression for our last step. They might say we "took the limit of Eq. 1.3 as Δt goes to zero." That is, we might have written Eq. 1.3 as

$$\lim_{\Delta t \to 0}\left(\frac{V_c(t + \Delta t) - V_c(t)}{\Delta t} + \frac{1}{RC} V_c(t) = 0\right). \quad (1.15)$$

Of course, we now can appreciate that Δt does not go all the way to zero. It just gets arbitrarily small, such that the discrete "chunking" of time is indistinguishable from a continuous flow of time in whatever context is appropriate. Well, it turns out that the limit

$$\lim_{\Delta t \to 0} \frac{V_c(t + \Delta t) - V_c(t)}{\Delta t} \tag{1.16}$$

occurs so often in the mathematics of continuous variables that we give ourselves an abbreviation, or a shorthand:

$$\lim_{\Delta t \to 0} \frac{V_c(t + \Delta t) - V_c(t)}{\Delta t} = \frac{dV_c(t)}{dt}. \tag{1.17}$$

This shorthand is called the derivative, as in "the derivative of V_c with respect to t." You may or may not remember from when you first learned derivatives that Eq. 1.16 was the formal definition given to you. We may therefore rewrite Eq. 1.3 using the shorthand

$$\frac{dV_c(t)}{dt} + \frac{1}{RC} V_c(t) = 0. \tag{1.18}$$

This is just a standard, first-order differential equation. The standard procedure here is to "guess" the solution Ae^{st}. Plugging this solution into Eq. 1.18 results in s being determined as $-1/RC$, and then we choose A to be V_0 in order to satisfy the initial conditions. The point of all this is that Eq. 1.18 does not spring out of a vacuum. Starting with Eq. 1.1, we took a very common-sense approach to solving a dynamical problem whose physics we understood pretty well. The approach represented by Eq. 1.18 takes for granted all of the insight that we gained by plodding through our discrete-time development. This is completely appropriate, as once the basics are understood it is important to streamline our methods as practical matter.

On a final note, we may interpret Eq. 1.18 in another way that makes its meaning jump off the page. We can write it as

$$\frac{dV_c(t)}{dt} = -\frac{1}{RC} V_c(t). \tag{1.19}$$

Putting this equation into words, we might say "The rate of change of the voltage across the capacitor is proportional to the voltage across it at any given time, and inversely proportional to the value of the RC product. That rate of change has the opposite sign of the voltage across the capacitor at a given time, so the magnitude of the voltage is always decreasing. The system comes to rest, which is to say, the rate of change of the capacitor voltage goes to zero, only when the voltage across the capacitor itself is zero." We see that this differential equation is indeed a very natural way to describe the time evolution of an RC circuit.

1.1.2 Higher-Order Systems

It turns out that the discretized development of Section 1.1.1 is readily extensible to higher-order systems. The first thing to do is to figure out the equivalent of Eq. 1.17 for higher-order derivatives. It is helpful to introduce additional notation; we often write the first derivative of a function $f(t)$ with respect to time as $f'(t)$. That is,

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{df(t)}{dt}. \tag{1.20}$$

Now we have a function $f'(t)$. We might ask, what is the time rate of change of this new function? You may remember that the answer is the second derivative of f with respect to t:

$$f''(t) = \frac{d}{dt}\frac{df(t)}{dt} = \frac{d^2 f}{dt^2}. \tag{1.21}$$

To figure out the equivalent of Eq. 1.17, we simply find the derivative of $f'(t)$,

$$f''(t) = \lim_{\Delta t \to 0} \frac{f'(t+\Delta t) - f'(t)}{\Delta t}, \tag{1.22}$$

and substitute the definition of $f'(t)$ from Eq. 1.20. Doing so yields

$$f''(t) = \lim_{\Delta t \to 0} \frac{f(t+2\cdot\Delta t) - 2f(t+\Delta t) + f(t)}{(\Delta t)^2}. \tag{1.23}$$

Repeating this procedure over and over again, we can get whatever order derivative we wish.

Higher-order derivatives come up quickly as we go beyond the complexity of the RC circuit in Figure 1.1. For example, consider the LC circuit in Figure 1.2. Proceeding in the same spirit that led to Eq. 1.1, we can write

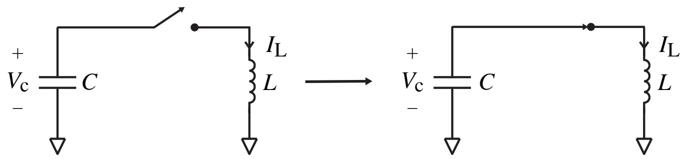

Figure 1.2 A simple LC circuit. The capacitor has an initial charge $Q_0 = CV_0$, and current begins to flow when the switch is closed at time $t = 0$.

$$CV_c(t+\Delta t) = CV_c(t) - I_L(t)\Delta t \tag{1.24}$$
$$I_L(t+\Delta t) = I_L(t) + \frac{1}{L}V_c(t)\Delta t.$$

One way to proceed from here is to solve the first equation for $I_L(t)$ in terms of $V_c(t)$ and $V_c(t+\Delta t)$, and then substitute for $I_L(t)$ and $I_L(t+\Delta t)$ in the second equation. Doing so causes our second-order derivative to appear right away:

$$\frac{V_c(t+2\cdot\Delta t) - 2V_c(t+\Delta t) + V_c(t)}{(\Delta t)^2} + \frac{1}{LC}V_c(t) = 0. \tag{1.25}$$

Now we proceed as before. We "guess" that $V_c(t)$ has the form $V_0 a^{n\cdot\Delta t}$, and are led to the quadratic characteristic equation in $a^{\Delta t}$:

$$(a^{\Delta t})^2 - 2(a^{\Delta t}) + 1 + \frac{(\Delta t)^2}{LC} = 0. \tag{1.26}$$

The quadratic formula readily provides us with possible values of $a^{\Delta t}$:

$$a^{\Delta t} = \frac{2 \pm \sqrt{4 - 4\left(1 + \frac{(\Delta t)^2}{LC}\right)}}{2} = 1 \pm j\frac{\Delta t}{\sqrt{LC}}. \tag{1.27}$$

As before, we take the log of both sides,

$$\ln a = \frac{1}{\Delta t}\ln\left(1 \pm j\frac{\Delta t}{\sqrt{LC}}\right), \tag{1.28}$$

only to encounter the log of a complex number. Dealing with this requires that we dust off a few important facts about complex numbers. The first is Euler's relation, which is

$$e^{j\theta} = \cos\theta + j\sin\theta. \tag{1.29}$$

The second fact is that any complex number $c + jd$ can be written in the polar form $re^{j\theta}$, where

$$r = \sqrt{c^2 + d^2} \tag{1.30}$$

and

$$\theta = \arctan\left(\frac{d}{c}\right). \tag{1.31}$$

The logarithm of this polar form is immediately apparent as

$$\ln(re^{j\theta}) = \ln r + j\theta. \tag{1.32}$$

Putting all of these facts together, we are free once again to pursue our original aim, which was solving for a. The argument of the logarithm in Eq. 1.28 becomes $re^{j\theta}$, where

$$r = \left(1 + \frac{(\Delta t)^2}{LC}\right)^{1/2} \tag{1.33}$$

and

$$\theta = \arctan\left(\frac{\Delta t}{\sqrt{LC}}\right). \tag{1.34}$$

So now Eq. 1.28 becomes

$$\ln a = \frac{1}{2 \cdot \Delta t} \ln\left(1 + \frac{(\Delta t)^2}{LC}\right) + \frac{j}{\Delta t} \arctan\left(\frac{\Delta t}{\sqrt{LC}}\right). \tag{1.35}$$

We now once again take a look at the consequences of a small Δt, this time noting that its smallness compared to $\sqrt{LC}$ is what counts. With the logarithmic term on the right side of Eq. 1.35, we do the same approximation that we did in Eq. 1.45. For the arctan term, we note that for $x \ll 1$, it can be shown that $\arctan x \approx x$. These approximations reduce Eq. 1.35 to

$$\ln a \approx \frac{1}{2}\frac{\Delta t}{LC} + \frac{j}{\sqrt{LC}}. \tag{1.36}$$

Here we notice one more thing: arbitrarily small Δt compared to $\sqrt{LC}$ makes for one further simplification, which is that

$$\ln a \approx \frac{j}{\sqrt{LC}}. \tag{1.37}$$

At the end of it all, we find that $a^{\Delta t} = e^{\pm j \cdot \Delta t/\sqrt{LC}}$, and therefore we can write the most general possible solution for V_c as

$$V_c(t) = Ae^{+jt/\sqrt{LC}} + Be^{-jt/\sqrt{LC}}. \tag{1.38}$$

In actual applications, A and B are determined by the initial conditions for V_c and I_L. To see this, we can write the general solution for I_L using Eq. 1.24. Now that we are confident of its meaning, we freely employ the shorthand for the derivative and rewrite Eq. 1.24 as

$$I_L(t) = -C\frac{dV_c}{dt} = -\frac{j}{\sqrt{LC}}Ae^{+jt/\sqrt{LC}} + \frac{j}{\sqrt{LC}}Be^{-jt/\sqrt{LC}}. \tag{1.39}$$