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1 VECTORS AND KINEMATICS

1.1 Introduction

Mechanics is at the heart of physics; its concepts are essential for understanding the world around us and phenomena on scales from atomic to cosmic. Concepts such as momentum, angular momentum, and energy play roles in practically every area of physics. The goal of this book is to help you acquire a deep understanding of the principles of mechanics.

The reason we start by discussing vectors and kinematics rather than plunging into dynamics is that we want to use these tools freely in discussing physical principles. Rather than interrupt the flow of discussion later, we are taking time now to ensure they are on hand when required.

1.2 Vectors

The topic of vectors provides a natural introduction to the role of mathematics in physics. By using vector notation, physical laws can often be written in compact and simple form. Modern vector notation was invented by a physicist, Willard Gibbs of Yale University, primarily to simplify the appearance of equations. For example, here is how Newton's second law appears in nineteenth century notation:

$$F_x = ma_x$$
$$F_y = ma_y$$
$$F_z = ma_z.$$

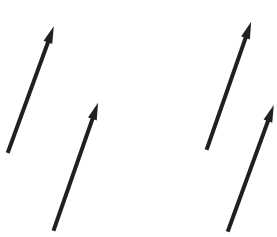
In vector notation, one simply writes

$$\mathbf{F} = m\mathbf{a},$$

where the bold face symbols **F** and **a** stand for vectors.

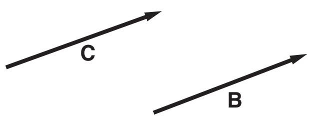
Our principal motivation for introducing vectors is to simplify the form of equations. However, as we shall see in Chapter 14, vectors have a much deeper significance. Vectors are closely related to the fundamental ideas of symmetry and their use can lead to valuable insights into the possible forms of unknown laws.

1.2.1 Definition of a Vector



Mathematicians think of a vector as a set of numbers accompanied by rules for how they change when the coordinate system is changed. For our purposes, a down to earth geometric definition will do: we can think of a vector as a *directed line segment*. We can represent a vector graphically by an arrow, showing both its scale length and its direction. Vectors are sometimes labeled by letters capped by an arrow, for instance $\vec{A}$, but we shall use the convention that a bold face letter, such as **A**, stands for a vector.

To describe a vector we must specify both its length and its direction. Unless indicated otherwise, we shall assume that parallel translation does not change a vector. Thus the arrows in the sketch all represent the same vector.



If two vectors have the same length and the same direction they are equal. The vectors **B** and **C** are equal:

$$\mathbf{B} = \mathbf{C}.$$

The magnitude or size of a vector is indicated by vertical bars or, if no confusion will occur, by using italics. For example, the magnitude of **A** is written $|\mathbf{A}|$, or simply A. If the length of **A** is $\sqrt{2}$, then $|\mathbf{A}| = A = \sqrt{2}$. Vectors can have physical dimensions, for example distance, velocity, acceleration, force, and momentum.

If the length of a vector is one unit, we call it a *unit vector*. A unit vector is labeled by a caret; the vector of unit length parallel to **A** is $\hat{\mathbf{A}}$. It follows that

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{A}$$

and conversely

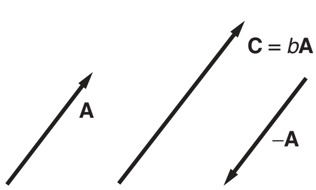
$$\mathbf{A} = A\hat{\mathbf{A}}.$$

The physical dimension of a vector is carried by its magnitude. Unit vectors are dimensionless.

1.3 The Algebra of Vectors

We will need to add, subtract, and multiply two vectors, and carry out some related operations. We will not attempt to divide two vectors since the need never arises, but to compensate for this omission, we will define two types of vector multiplication, both of which turn out to be quite useful. Here is a summary of the basic algebra of vectors.

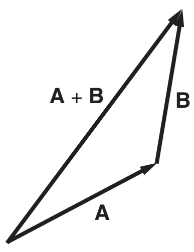
1.3.1 Multiplying a Vector by a Scalar



If we multiply **A** by a simple scalar, that is, by a simple number b, the result is a new vector $\mathbf{C} = b\mathbf{A}$. If $b > 0$ the vector **C** is parallel to **A**, and its magnitude is b times greater. Thus $\hat{\mathbf{C}} = \hat{\mathbf{A}}$, and $C = bA$.

If $b < 0$, then $\mathbf{C} = b\mathbf{A}$ is opposite in direction (antiparallel) to **A**, and its magnitude is $C = |b|\,A$.

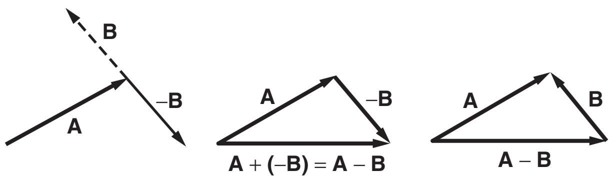
1.3.2 Adding Vectors



Addition of two vectors has the simple geometrical interpretation shown by the drawing. The rule is: to add **B** to **A**, place the tail of **B** at the head of **A** by parallel translation of **B**. The sum is a vector from the tail of **A** to the head of **B**.

1.3.3 Subtracting Vectors

Because $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$, to subtract **B** from **A** we can simply multiply **B** by –1 and then add. The sketch shows how.



An equivalent way to construct **A** – **B** is to place the *head* of **B** at the *head* of **A**. Then **A** – **B** extends from the *tail* of **A** to the *tail* of **B**, as shown in the drawing.

1.3.4 Algebraic Properties of Vectors

It is not difficult to prove the following:

Commutative law

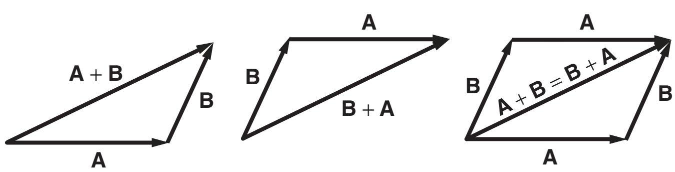
$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}.$$

Associative law

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$
$$c(d\mathbf{A}) = (cd)\mathbf{A}.$$

Distributive law

$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$$
$$(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}.$$



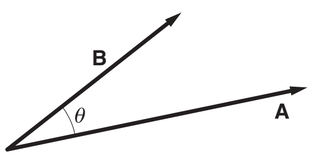
The sketch shows a geometrical proof of the commutative law **A** + **B** = **B** + **A**; try to cook up your own proofs of the others.

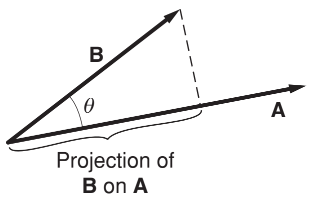
1.4 Multiplying Vectors

Multiplying one vector by another could produce a vector, a scalar, or some other quantity. The choice is up to us. It turns out that two types of vector multiplication are useful in physics.

1.4.1 Scalar Product ("Dot Product")

The first type of multiplication is called the *scalar* product because the result of the multiplication is a scalar. The scalar product is an operation





that combines vectors to form a scalar. The scalar product of **A** and **B** is written as $\mathbf{A}\cdot\mathbf{B}$, therefore often called the *dot* product. $\mathbf{A}\cdot\mathbf{B}$ (referred to as "A dot B") is defined by

$$\mathbf{A}\cdot\mathbf{B} \equiv AB\cos\theta.$$

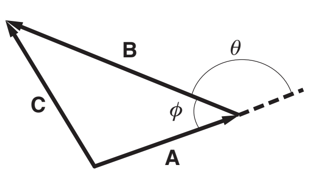
Here θ is the angle between **A** and **B** when they are drawn tail to tail. Because $B\cos\theta$ is the projection of **B** along the direction of **A**, it follows that

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= A \text{ times the projection of } \mathbf{B} \text{ on } \mathbf{A}\\ &= B \text{ times the projection of } \mathbf{A} \text{ on } \mathbf{B}.\end{aligned}$$

Note that $\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2 = A^2$. Also, $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$; the order does not change the value. We say that the dot product is *commutative*.

If either **A** or **B** is zero, their dot product is zero. However, because $\cos\pi/2 = 0$ the dot product of two non-zero vectors is nevertheless zero if the vectors happen to be perpendicular.

A great deal of elementary trigonometry follows from the properties of vectors. Here is an almost trivial proof of the law of cosines using the dot product.



Example 1.1 The Law of Cosines

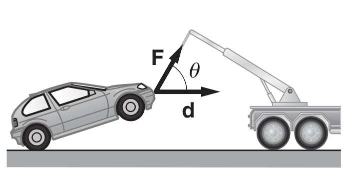
The law of cosines relates the lengths of three sides of a triangle to the cosine of one of its angles. Following the notation of the drawing, the law of cosines is

$$C^2 = A^2 + B^2 - 2AB\cos\phi.$$

The law can be proved by a variety of trigonometric or geometric constructions, but none is so simple and elegant as the vector proof, which merely involves squaring the sum of two vectors.

$$\begin{aligned}\mathbf{C} &= \mathbf{A}+\mathbf{B}\\ \mathbf{C}\cdot\mathbf{C} &= (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})\\ &= \mathbf{A}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B}+2(\mathbf{A}\cdot\mathbf{B})\\ C^2 &= A^2+B^2+2AB\cos\theta.\end{aligned}$$

Recognizing that $\cos\phi = -\cos\theta$ completes the proof.



Example 1.2 Work and the Dot Product

The dot product has an important physical application in describing the work done by a force. As you may already know, the work W done on an object by a force F is defined to be the product of the length of the displacement d and the component of F along the direction of displacement. If the force is applied at an angle θ with respect to the displacement, as shown in the sketch,

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then

$$W = (F\cos\theta)d.$$

Assuming that force and displacement can both be written as vectors, then

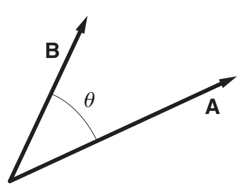
$$W = \mathbf{F}\cdot\mathbf{d}.$$

1.4.2 Vector Product ("Cross Product")

The second type of product useful in physics is the *vector* product, in which two vectors **A** and **B** are combined to form a third vector **C**. The symbol for vector product is a cross, so it is often called the *cross* product:

$$\mathbf{C} = \mathbf{A}\times\mathbf{B}.$$

The vector product is more complicated than the scalar product because we have to specify both the magnitude and direction of the vector $\mathbf{A}\times\mathbf{B}$ (called "A cross B"). The magnitude is defined as follows: if



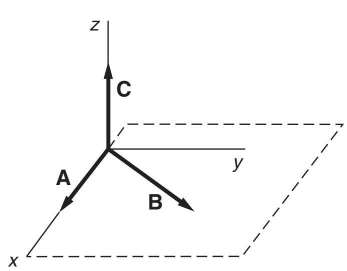
$$\mathbf{C} = \mathbf{A}\times\mathbf{B}$$

then

$$C = AB\sin\theta$$

where θ is the angle between **A** and **B** when they are drawn tail to tail.

To eliminate ambiguity, θ is always taken as the angle smaller than π. Even if neither vector is zero, their vector product is zero if $\theta = 0$ or π, the situation where the vectors are parallel or antiparallel. It follows that



$$\mathbf{A}\times\mathbf{A} = 0$$

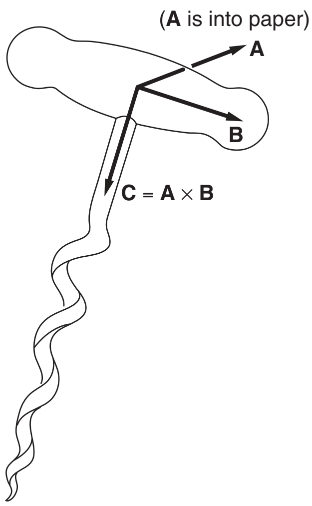
for any vector **A**.

Two vectors **A** and **B** drawn tail to tail determine a plane. Any plane can be drawn through **A**. Simply rotate it until it also contains **B**.

We define the direction of **C** to be perpendicular to the plane of **A** and **B**. The three vectors **A**, **B**, and **C** form what is called a *right-hand triple*. Imagine a right-hand coordinate system with **A** and **B** in the x–y plane as shown in the sketch.

A lies on the x axis and **B** lies toward the y axis. When **A**, **B**, and **C** form a right-hand triple, then **C** lies along the positive z axis. We shall always use right-hand coordinate systems such as the one shown.

Here is another way to determine the direction of the cross product. Think of a right-hand screw with the axis perpendicular to **A** and **B**.

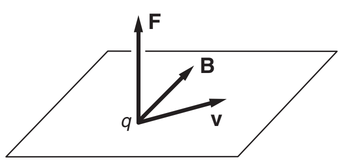


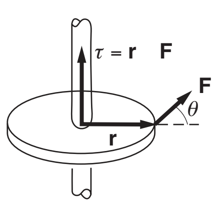
If we rotate it in the direction that swings **A** into **B**, then **C** lies in the direction the screw advances. (Warning: be sure not to use a left-hand screw. Fortunately, they are rare, with hot water faucets among the chief offenders. Your honest everyday wood screw is right-handed.)

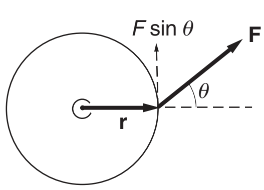
A result of our definition of the cross product is that

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}.$$

Here we have a case in which the order of multiplication is important. The vector product is *not* commutative. Since reversing the order reverses the sign, it is *anticommutative*.







Top view

Example 1.3 Examples of the Vector Product in Physics

The vector product has a multitude of applications in physics. For instance, if you have learned about the interaction of a charged particle with a magnetic field, you know that the force is proportional to the charge q, the magnetic field B, and the velocity of the particle v. The force varies as the sine of the angle between v and B, and is perpendicular to the plane formed by v and B, in the direction indicated.

All these rules are combined in the one equation

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

Another application is the definition of torque, which we shall develop in Chapter 7. For now we simply mention in passing that the torque vector $\boldsymbol{\tau}$ is defined by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F},$$

where **r** is a vector from the axis about which the torque is evaluated to the point of application of the force **F**. This definition is consistent with the familiar idea that torque is a measure of the ability of an applied force to produce a twist. Note that a large force directed parallel to **r** produces no twist; it merely pulls. Only $F \sin\theta$, the component of force perpendicular to **r**, produces a torque.

Imagine that we are pushing open a garden gate, where the axis of rotation is a vertical line through the hinges. When we push the gate open, we instinctively apply force in such a way as to make **F** closely perpendicular to **r** , to maximize the torque. Because the torque increases as the lever arm gets larger, we push at the edge of the gate, as far from the hinge line as possible.

As you will see in Chapter 7, the natural direction of $\boldsymbol{\tau}$ is along the axis of the rotation that the torque tends to produce. All these ideas are summarized in a nutshell by the simple equation $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

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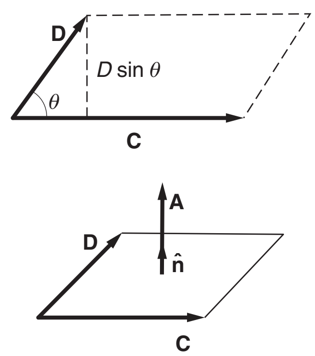


Example 1.4 Area as a Vector

We can use the cross product to describe an area. Usually one thinks of area in terms of magnitude only. However, many applications in physics require that we also specify the orientation of the area. For example, if we wish to calculate the rate at which water in a stream flows through a wire loop of given area, it obviously makes a difference whether the plane of the loop is perpendicular or parallel to the flow. (If parallel, the flow through the loop is zero.) Here is how the vector product accomplishes this:

Consider the area of a quadrilateral formed by two vectors **C** and **D**. The area A of the parallelogram is given by

$$\begin{aligned} A &= \text{base} \times \text{height} \\ &= CD\sin\theta \\ &= |\mathbf{C}\times\mathbf{D}|\,. \end{aligned}$$

The magnitude of the cross product gives us the area of the parallelogram, but how can we assign a direction to the area? In the plane of the parallelogram we can draw an infinite number of vectors pointing every which-way, so none of these vectors stands out uniquely. The only unique preferred direction is the *normal* to the plane, specified by a unit vector $\hat{\mathbf{n}}$. We therefore take the vector **A** describing the area as parallel to $\hat{\mathbf{n}}$. The magnitude and direction of **A** are then given compactly by the cross product

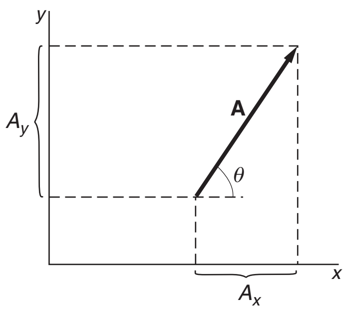
$$\mathbf{A} = \mathbf{C}\times\mathbf{D}.$$

A minor ambiguity remains, because $\hat{\mathbf{n}}$ can point out from either side of the area. We could just as well have defined the area by $\mathbf{A} = \mathbf{D}\times\mathbf{C} = -\mathbf{C}\times\mathbf{D}$, as long as we are consistent once the choice is made.

1.5 Components of a Vector

The fact that we have discussed vectors without introducing a particular coordinate system shows why vectors are so useful; vector operations are defined independently of any particular coordinate system. However, eventually we have to translate our results from the abstract to the concrete, and at this point we have to choose a coordinate system in which to work.

The combination of algebra and geometry, called *analytic geometry*, is a powerful tool that we shall use in many calculations. Analytic geometry has a consistent procedure for describing geometrical objects by a set of numbers, greatly easing the task of performing quantitative calculations. With its aid, students still in school can routinely solve problems that would have taxed the ancient Greek geometer Euclid. Analytic geometry was developed as a complete subject in the first half of the seventeenth century by the French mathematician René Descartes, and independently by his contemporary Pierre Fermat.



For simplicity, let us first restrict ourselves to a two-dimensional system, the familiar x–y plane. The diagram shows a vector **A** in the x–y plane.

The projections of **A** along the x and y coordinate axes are called the *components* of **A**, A_x and A_y, respectively. The magnitude of **A** is $A = \sqrt{A_x{}^2 + A_y{}^2}$, and the direction of **A** makes an angle $\theta = \arctan\,(A_y/A_x)$ with the x axis.

Since its components define a vector, we can specify a vector entirely by its components. Thus

$$\mathbf{A} = (A_x, A_y)$$

or, more generally, in three dimensions,

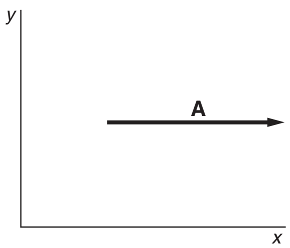
$$\mathbf{A} = (A_x, A_y, A_z).$$

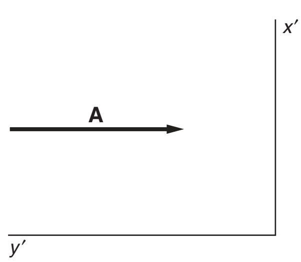
Prove for yourself that $A = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}$.

If two vectors are equal $\mathbf{A} = \mathbf{B}$, then in the same coordinate system their corresponding components are equal.

$$A_x = B_x \quad A_y = B_y \quad A_z = B_z.$$

The single vector equation $\mathbf{A} = \mathbf{B}$ symbolically represents three scalar equations.





The vector **A** has a meaning independent of any coordinate system. However, the components of **A** depend on the coordinate system being used. To illustrate this, here is a vector **A** drawn in two different coordinate systems.

In the first case,

$$\mathbf{A} = (A, 0) \quad (x, y \text{ system}),$$

while in the second

$$\mathbf{A} = (0, -A) \quad (x', y' \text{ system}).$$

All vector operations can be written as equations for components. For instance, multiplication by a scalar is written

$$c\mathbf{A} = (cA_x, cA_y, cA_z).$$

The law for vector addition is

$$\mathbf{A} + \mathbf{B} = (A_x + B_x,\ A_y + B_y,\ A_z + B_z).$$

By writing **A** and **B** as the sums of vectors along each of the coordinate axes, you can verify that

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z.$$

We shall defer evaluating the cross product until the next section.

Example 1.5 Vector Algebra

Let

$$\mathbf{A} = (3,\ 5,\ -7)$$
$$\mathbf{B} = (2,\ 7,\ 1).$$

Find $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, A, B, $\mathbf{A} \cdot \mathbf{B}$, and the cosine of the angle between $\mathbf{A}$ and $\mathbf{B}$.

$$\begin{aligned}
\mathbf{A} + \mathbf{B} &= (3 + 2,\ 5 + 7,\ -7 + 1) \\
&= (5,\ 12,\ -6) \\
\mathbf{A} - \mathbf{B} &= (3 - 2,\ 5 - 7,\ -7 - 1) \\
&= (1,\ -2,\ -8) \\
A &= \sqrt{(3^2 + 5^2 + 7^2)} \\
&= \sqrt{83} \\
&\approx 9.11 \\
B &= \sqrt{(2^2 + 7^2 + 1^2)} \\
&= \sqrt{54} \\
&\approx 7.35 \\
\mathbf{A} \cdot \mathbf{B} &= 3 \times 2 + 5 \times 7 - 7 \times 1 \\
&= 34 \\
\cos(\mathbf{A}, \mathbf{B}) &= \frac{\mathbf{A} \cdot \mathbf{B}}{AB} \approx \frac{34}{(9.11)(7.35)} \approx 0.508.
\end{aligned}$$

Example 1.6 Constructing a Vector Perpendicular to a Given Vector

The problem is to find a unit vector lying in the x–y plane that is perpendicular to the vector $\mathbf{A} = (3,\ 5,\ 1)$.

A vector $\mathbf{B}$ in the x–y plane has components $(B_x,\ B_y)$. For $\mathbf{B}$ to be perpendicular to $\mathbf{A}$, we must have $\mathbf{A} \cdot \mathbf{B} = 0$:

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= 3B_x + 5B_y \\
&= 0.
\end{aligned}$$

Hence $B_y = -\frac{3}{5}B_x$. For $\mathbf{B}$ to be a unit vector, $B_x{}^2 + B_y{}^2 = 1$. Combining these gives

$$B_x{}^2 + \frac{9}{25}B_x{}^2 = 1,$$

or

$$\begin{aligned}
B_x &= \sqrt{\frac{25}{34}} \\
&\approx \pm 0.858 \\
B_y &= -\tfrac{3}{5}B_x \\
&\approx \mp 0.515.
\end{aligned}$$