ENCYCLOPEDIA OF SPECIAL FUNCTIONS: THE ASKEY–BATEMAN PROJECT

Volume 1: Univariate Orthogonal Polynomials

This is the first of three volumes that form the *Encyclopedia of Special Functions*, an extensive update of the Bateman Manuscript Project.

Volume 1 contains most of the material on orthogonal polynomials, from the classical orthogonal polynomials of Hermite, Laguerre and Jacobi to the Askey–Wilson polynomials, which are the most general basic hypergeometric orthogonal polynomials. Separate chapters cover orthogonal polynomials on the unit circle, zeros of orthogonal polynomials and matrix orthogonal polynomials, with detailed results about matrix-valued Jacobi polynomials. A chapter on moment problems provides many examples of indeterminate moment problems. A thorough bibliography rounds off what will be an essential reference.

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Volume 1: Univariate Orthogonal Polynomials

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Contents

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Preface

In the early 1970s, Richard Askey proposed to update the Bateman Project (Erdélyi et al., 1953a–1955). The rationale was that, at the time, he believed that the one-variable theory was more or less complete and it was time to update the Bateman Project and incorporate the results missed by it, or discovered after its publication. At the time this was indeed how many people felt. The big surprise came in 1977 when Wilson, under Askey's guidance, discovered what are now known as the Wilson polynomials, which are today familiar across mathematics. This was followed by many q-polynomials and cumulated in the Askey–Wilson polynomials. At the same time, the q-ultraspherical polynomials were identified and a new scheme of orthogonal polynomials emerged. Some other polynomials, which do not share the properties of classical polynomials, were discovered. These include the Pollaczek polynomials (Pollaczek, 1956) and the random walk polynomials of Askey and Ismail (1984) together with their q-analogues, as well as the Lommel-type polynomials of Wimp (1985) and Ismail (1998). Then the associated polynomials of the classical orthogonal polynomials and their q-analogues attracted a lot of interest and this also led to the sieved polynomials (Al-Salam, Allaway, and Askey, 1984). At the same time, a combinatorial theory of orthogonal polynomials was developed by many notable mathematicians, spearheaded by the French school of Foata and Viennot (Cartier and Foata, 1969; Viennot, 1983). These remarks outline the developments in special systems of orthogonal polynomials. They indirectly led to developments in the theory of continued fractions and other special functions. We also started to see developments in the theory of multivariate orthogonal polynomials and their q-analogues, together with parallel developments in multivariate statistics. The emergence of quantum groups then led to developments in orthogonal polynomials and special functions (Koornwinder, 1990). Since the 1990s we have seen a strong interaction between integrable systems and special functions and orthogonal polynomials. The Riemann–Hilbert approach of Fokas, Its, and Kitaev, combined with the nonlinear steepest descent method of Deift and Zhou (Deift, 1999), revolutionized the asymptotic theory of orthogonal polynomials.

This volume is the first in a series of volumes which form an *Encyclopedia of Special Functions*. This volume contains most of the material on univariate orthogonal polynomials. The combinatorial results on orthogonal polynomials, as well as related topics in continued fractions, will be in a later volume, which will also contain the material on exceptional orthogonal polynomials, as well as a coverage of multiple orthogonal polynomials.

The series will be edited jointly with Walter Van Assche, who will also contribute some future chapters. Many people have worked on this volume. The chapter authors are indicated at the beginning of each chapter and we are grateful to all of them for their contributions. Mourad Ismail wrote the remaining chapters. Erik Koelink and Tom Koornwinder wrote the material on addition theorems in Section 2.7.

The subject of orthogonal polynomials is still an active research area and we have no doubt that this will continue for years to come. Our beloved Mizan Rahman passed away on January 5, 2015, Martin Muldoon on August 1, 2019 and Richard Askey on October 9, 2019. We are sure they are with us in spirit and we are sorry that they did not live to see the completed project.

I gratefully acknowledge the help of Denise Marks of the University of South Florida in putting the material together. David Tranah of Cambridge University Press has been very encouraging and patient over the years; thanks David. The subject and this project owe a great deal to the mathematical contributions and leadership of George Andrews and Richard Askey. I would like to thank the King Saud University for generous research support during the last five years through the Distinguished Scientist Fellowship Program. This part-time affiliation with King Saud University has been very beneficial. Last but not least, I thank the University of Central Florida for my research appointment which gave me ample time to concentrate on research and writing.

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