1
Preliminaries

In this chapter we define the special functions used in this volume and state the properties relevant to the treatment of orthogonal polynomials. We also state a few facts from complex analysis used in the later parts.

1.1 Analytic Facts

Two important special cases of the Lagrange expansion are

$$e^{\alpha z} = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+n)^{n-1}}{n!} w^n, \quad w = ze^{-z}, \tag{1.1.1}$$

$$(1+z)^{\alpha} = 1 + \alpha \sum_{n=1}^{\infty} \binom{\alpha+\beta n-1}{n-1} \frac{w^n}{n}, \quad w = z(1+z)^{-\beta}. \tag{1.1.2}$$

The Perron–Stieltjes inversion formula (see Stone, 1932, Lemma 5.2) is

$$F(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}, \quad z \notin \mathbb{R} \tag{1.1.3}$$

if and only if

$$\mu(t) - \mu(s) = \lim_{\epsilon \to 0^+} \int_s^t \frac{F(x-i\epsilon) - F(x+i\epsilon)}{2\pi i}\, dx. \tag{1.1.4}$$

Formula (1.1.4) shows that the absolutely continuous component of μ is

$$\mu'(x) = \left[F\left(x - i0^+\right) - F\left(x + i0^+\right)\right]/(2\pi i). \tag{1.1.5}$$

Here μ is normalized by $\mu(x) = \left[\mu\left(x+0^+\right) + \mu\left(x+0^-\right)\right]/2$.

Definition 1.1.1 Let f be an entire function. The maximum modulus is

$$M(r; f) := \sup\left\{|f(z)| : |z| \le r\right\}, \quad r > 0. \tag{1.1.6}$$

The order $\rho(f)$ of f is defined by

$$\rho(f) := \limsup_{r\to\infty} \frac{\ln\ln M(r, f)}{\ln r}. \tag{1.1.7}$$

Theorem 1.1.2 (Boas, 1954) *If $\rho(f)$ is finite and is not equal to a positive integer, then f has infinitely many zeros.*

If f has finite order, its type σ is

$$\sigma = \inf\{K : M(r) < \exp(Kr^{\rho})\}. \tag{1.1.8}$$

The Phragmén–Lindelöf indicator of an entire function of finite order and type is

$$h(\theta) = \limsup_{r\to\infty} \frac{\ln\left|f(re^{i\theta})\right|}{r^{\rho}}. \tag{1.1.9}$$

Theorem 1.1.3 *Given two differential equations in the form*

$$\frac{d^2u}{dz^2} + f(z)u(z) = 0, \qquad \frac{d^2v}{dz^2} + g(z)v(z) = 0,$$

then $y = uv$ satisfies

$$\frac{d}{dz}\left\{\frac{y''' + 2(f+g)y' + (f'+g')y}{f-g}\right\} + (f-g)y = 0 \quad \text{if } f \neq g, \tag{1.1.10}$$

$$y''' + 4fy' + 2f'y = 0 \quad \text{if } f = g. \tag{1.1.11}$$

Watson (1944, §5.4), attributes Theorem 1.1.3 to P. Appell.

Lemma 1.1.4 *Let $y = y(x)$ satisfy the differential equation*

$$\phi(x)y''(x) + y(x) = 0, \quad a < x < b, \tag{1.1.12}$$

where $\phi(x) > 0$, and $\phi'(x)$ is positive (negative) and continuous on (a, b). Then the successive relative maxima of $|y|$ increase (decrease) with x in (a, b) if ϕ increases (decreases) on (a, b).

The Wronskian of f and g is

$$W(f, g) := fg' - gf' = \det\begin{pmatrix} f & g \\ f' & g' \end{pmatrix}. \tag{1.1.13}$$

1.2 Hypergeometric Functions

Standard references in the area of special functions are Andrews, Askey, and Roy (1999), Bailey (1935), Rainville (1960), Erdélyi et al. (1953a), Slater (1966), and the real classic Whittaker and Watson (1927). The **shifted factorial** is

$$(a)_0 := 1, \quad (a)_n := a(a+1)\cdots(a+n-1), \quad n > 0, \tag{1.2.1}$$

so that

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{1.2.2}$$

Note that (1.2.2) is meaningful for any complex n, when $a + n$ is not a pole of the gamma function. The **multishifted factorial** is

$$(a_1, \ldots, a_m)_n = \prod_{j=1}^{m} \left(a_j\right)_n .$$

The difference operators are

$$\begin{aligned} \Delta f(x) &= (\Delta f)(x) := f(x+1) - f(x), \\ \nabla f(x) &= (\nabla f)(x) := f(x) - f(x-1). \end{aligned} \tag{1.2.3}$$

We will also use the symmetric difference operator

$$\left(\tilde{\Delta}_h\right) f(x) = [f(x + h/2) - f(x - h/2)]/h. \tag{1.2.4}$$

It is clear that $\tilde{\Delta}_h$ is a discrete analogue of the derivative. Another divided difference operator is the **Wilson operator**,

$$(\mathcal{W} f)(x) = \frac{\tilde{f}(y + i/2) - \tilde{f}(y - i/2)}{\tilde{e}(y + i/2) - \tilde{e}(y - i/2)}, \tag{1.2.5}$$

where

$$y = \sqrt{x}, \quad \tilde{e}(y) = x, \quad \text{and} \quad \tilde{f}(y) := f(x). \tag{1.2.6}$$

It is easy to see that $\tilde{e}(y + i/2) - \tilde{e}(y - i/2) = 2i\sqrt{x}$. It is a fact that

$$\mathcal{W}\psi_n(x; a) = n\psi_{n-1}(x; a + 1/2), \tag{1.2.7}$$

where

$$\psi_n(x; a) := \left(a + i\sqrt{x}\right)_n \left(a - i\sqrt{x}\right)_n . \tag{1.2.8}$$

A hypergeometric series is

$${}_rF_s\left(\left.\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\right| z\right) = {}_rF_s\left(a_1, \ldots, a_r; b_1, \ldots, b_s; z\right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r)_n}{(b_1, \ldots, b_s)_n} \frac{z^n}{n!}. \tag{1.2.9}$$

If one of the numerator parameters is a negative integer, say $-k$, then the series (1.2.9) becomes a finite sum, $0 \le n \le k$, and the ${}_rF_s$ series is called **terminating**. As a function of z, the nonterminating series is an entire function if $r \le s$, and it is analytic in $\{|z| < 1\}$ if $r = s + 1$. The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ (Erdélyi et al., 1953a, §2.1) satisfies the **hypergeometric differential equation**

$$z(1 - z)\frac{d^2y}{dz^2} + [c - (a + b + 1)z]\frac{dy}{dz} - aby = 0, \tag{1.2.10}$$

and has the **Euler integral representation**

$${}_2F_1\left(a, b; c; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a}\, dt, \tag{1.2.11}$$

for $\operatorname{Re} b > 0$, $\operatorname{Re}(c - b) > 0$.

The **confluent hypergeometric function** (Erdélyi et al., 1953a, §6.1)

$$\Phi(a, c; z) := {}_1F_1(a; c; z) \tag{1.2.12}$$

satisfies the differential equation

$$z\frac{d^2y}{dz^2} + (c - z)\frac{dy}{dz} - ay = 0, \tag{1.2.13}$$

and $\lim_{b\to\infty} {}_2F_1(a, b; c; z/b) = {}_1F_1(a; c; z)$. The Tricomi Ψ function is a second linearly independent solution of (1.2.13) and is defined by (Erdélyi et al., 1953a, §6.5)

$$\Psi(a, c; x) := \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}\Phi(a-c+1, 2-c; x). \tag{1.2.14}$$

The function Ψ has the integral presentation (Erdélyi et al., 1953b, §6.5)

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}t^{a-1}(1+t)^{c-a-1}\,dt, \quad \operatorname{Re} a > 0,\ \operatorname{Re} x > 0. \tag{1.2.15}$$

The Bessel function J_ν and the modified Bessel function I_ν (Watson, 1944) are

$$J_\nu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{\nu+2n}}{\Gamma(n+\nu+1)n!}, \tag{1.2.16}$$

$$I_\nu(z) = e^{-i\pi\nu/2}J_\nu\left(ze^{i\pi/2}\right) = \sum_{n=0}^{\infty}\frac{(z/2)^{\nu+2n}}{\Gamma(n+\nu+1)n!}. \tag{1.2.17}$$

Observe the special cases

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z, \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z. \tag{1.2.18}$$

The Bessel functions satisfy the recurrence relation

$$\frac{2\nu}{z}J_\nu(z) = J_{\nu+1}(z) + J_{\nu-1}(z). \tag{1.2.19}$$

The Bessel functions J_ν and $J_{-\nu}$ satisfy the **Bessel differential equation**

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \nu^2\right)y = 0. \tag{1.2.20}$$

Another solution, $Y_\nu(z)$, to both (1.2.19) and (1.2.20) is

$$\begin{aligned} Y_\nu(z) &= \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}, \quad \nu \neq 0, \pm1, \pm2, \ldots, \\ Y_n(z) &= \lim_{\nu\to n} Y_\nu(z), \quad n = 0, \pm1, \pm2, \ldots. \end{aligned} \tag{1.2.21}$$

The functions $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent solutions of (1.2.20).

The function I_ν satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - \left(x^2 + \nu^2\right) y = 0, \tag{1.2.22}$$

whose second solution is

$$\begin{aligned} K_\nu(x) &= \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}, \quad \nu \neq 0, \pm 1, \pm 2, \dots, \\ K_n(x) &= \lim_{\nu\to n} K_\nu(x), \quad n = 0, \pm 1, \pm 2, \dots. \end{aligned} \tag{1.2.23}$$

We also have the recursion relations

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x), \quad K_{\nu+1}(x) - K_{\nu-1}(x) = \frac{2\nu}{x} K_\nu(x). \tag{1.2.24}$$

Two important integrals are **Sonine's first integral**,

$$J_{\alpha+\beta+1}(z) = \frac{2^{-\beta} z^{\beta+1}}{\Gamma(\beta+1)} \int_0^1 x^{2\beta+1} \left(1 - x^2\right)^{\alpha/2} J_\alpha\left(z\sqrt{1 - x^2}\right) dx, \tag{1.2.25}$$

for $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$, and **Sonine's second integral**,

$$x^\mu y^\nu \frac{J_{\mu+\nu+1}\left(\sqrt{x^2 + y^2}\right)}{(x^2 + y^2)^{(\mu+\nu+1)/2}} = \int_0^{\pi/2} J_\mu(x \sin\theta) J_\nu(y\cos\theta) \sin^{\mu+1}\theta \sin^{\nu+1}\theta \, d\theta \tag{1.2.26}$$

(Andrews, Askey, and Roy, 1999, Theorem 4.11.1).

Theorem 1.2.1 *When $\nu > -1$, the function $z^{-\nu} J_\nu(z)$ has only real and simple zeros. Furthermore, the positive (negative) zeros of $J_\nu(z)$ and $J_{\nu+1}(z)$ interlace for $\nu > -1$.*

We shall denote the positive zeros of $J_\nu(z)$ by $(j_{\nu,k})_k$, that is,

$$0 < j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \cdots. \tag{1.2.27}$$

The Bessel functions satisfy the differential recurrence relations (Watson, 1944)

$$zJ'_\nu(z) = \nu J_\nu(z) - zJ_{\nu+1}(z), \tag{1.2.28}$$

$$zY'_\nu(z) = \nu Y_\nu(z) - zY_{\nu+1}(z), \tag{1.2.29}$$

$$zI'_\nu(z) = zI_{\nu+1}(z) + \nu I_\nu(z), \tag{1.2.30}$$

$$zK'_\nu(z) = \nu K_\nu(z) - zK_{\nu+1}(z). \tag{1.2.31}$$

The Bessel functions are special cases of ${}_1F_1$ since (Erdélyi et al., 1953a, §6.9.1)

$$e^{-iz} {}_1F_1(\nu + 1/2; 2\nu + 1; 2iz) = \Gamma(\nu + 1)(z/2)^{-\nu} J_\nu(z). \tag{1.2.32}$$

The Airy function $A(x)$ is a combination of

$$k(x) := \frac{\pi}{3}\left(\frac{x}{3}\right)^{\frac{1}{2}} J_{-1/3}\left(2(x/3)^{3/2}\right) = \frac{\pi}{3}\sum_{n=0}^{\infty}\frac{(-x/3)^{3n}}{n!\Gamma(n+2/3)},$$
$$\ell(x) := \frac{\pi}{3}\left(\frac{x}{3}\right)^{\frac{1}{2}} J_{1/3}\left(2(x/3)^{3/2}\right) = \frac{\pi}{9}x\sum_{n=0}^{\infty}\frac{(-x/3)^{3n}}{n!\Gamma(n+4/3)}. \tag{1.2.33}$$

Indeed $\{k(x), \ell(x)\}$ is a basis of solutions of the Airy equation

$$\frac{d^2y}{dx^2} + \frac{1}{3}xy = 0. \tag{1.2.34}$$

The only solution of (1.2.34) which is bounded as $x \to -\infty$ is $k(x) + \ell(x)$. Set

$$A(x) := k(x) + \ell(x). \tag{1.2.35}$$

The function $A(x)$ is called the Airy function and has the asymptotic behavior

$$A(x) = \frac{\sqrt{\pi}}{2\,3^{1/4}}|x|^{-\frac{1}{4}}\exp\left\{-2(|x|/3)^{3/2}\right\}(1+o(1)), \quad x \to -\infty. \tag{1.2.36}$$

Nowadays it is more common to use the Airy function $\mathrm{Ai}(x)$ which is a solution of $y'' = xy$ that remains bounded as $x \to \infty$. The relation with (1.2.35) is $A(x) = 3^{-1/3}\pi\mathrm{Ai}(-3^{-1/3}x)$. The Airy function plays an important role in the theory of orthogonal polynomials with exponential weights, random matrix theory, as well as other parts of mathematical physics and applied mathematics. The function $A(x)$ is positive on $(-\infty, 0)$ and has only positive simple zeros. We shall denote the zeros of $A(x)$ by

$$0 < i_1 < i_2 < \cdots. \tag{1.2.37}$$

The Appell functions generalize the hypergeometric function to two variables. They are defined by (Appell and Kampé de Fériet, 1926; Erdélyi et al., 1953a)

$$F_1(a; b, b'; c; x, y) = \sum_{m,n=0}^{\infty}\frac{(a)_{n+m}(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n, \tag{1.2.38}$$

$$F_2(a; b, b'; c, c'; x, y) = \sum_{m,n=0}^{\infty}\frac{(a)_{n+m}(b)_m(b')_n}{(c)_m(c')_n m!n!}x^m y^n, \tag{1.2.39}$$

$$F_3(a, a'; b, b'; c; x, y) = \sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n, \tag{1.2.40}$$

$$F_4(a, b; c, c'; x, y) = \sum_{m,n=0}^{\infty}\frac{(a)_{n+m}(b)_{m+n}}{(c)_m(c')_n m!n!}x^m y^n. \tag{1.2.41}$$

The complete elliptic integrals of the first and second kinds are (Erdélyi et al., 1953b)

$$\mathbf{K} = \mathbf{K}(k) = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \tag{1.2.42}$$

$$\mathbf{E} = \mathbf{E}(k) = \int_0^1 \sqrt{\frac{1-k^2u^2}{1-u^2}}\, du, \tag{1.2.43}$$

respectively. One has

$$\mathbf{K}(k) = \frac{\pi}{2}\, {}_2F_1\left(1/2, 1/2; 1; k^2\right), \tag{1.2.44}$$

$$\mathbf{E}(k) = \frac{\pi}{2}\, {}_2F_1\left(-1/2, 1/2; 1; k^2\right). \tag{1.2.45}$$

We refer to k as the modulus, while the complementary modulus k' is

$$k' = \left(1-k^2\right)^{1/2}. \tag{1.2.46}$$

1.3 Summation Theorems and Transformations

In the shifted factorial notation the binomial theorem is

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}, \quad |z| < 1. \tag{1.3.1}$$

If a is a negative integer then the sum is finite and gives the familiar binomial formula

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k z^k = (1-z)^n.$$

The **Gauss sum** is

$${}_2F_1\left(\left.\begin{matrix} a, b \\ c \end{matrix}\,\right|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \operatorname{Re}\{c-a-b\} > 0. \tag{1.3.2}$$

The terminating version of (1.3.2) is the **Chu–Vandermonde sum**

$${}_2F_1\left(\left.\begin{matrix} -n, b \\ c \end{matrix}\,\right|\, 1\right) = \frac{(c-b)_n}{(c)_n}. \tag{1.3.3}$$

A hypergeometric series (1.2.9) is called **balanced** if $r = s + 1$ and

$$1 + \sum_{k=1}^{s+1} a_k = \sum_{k=1}^{s} b_k. \tag{1.3.4}$$

The sum of a terminating balanced ${}_3F_2$ of unit argument is the **Pfaff–Saalschütz theorem**

$${}_3F_2\left(\left.\begin{matrix} -n, a, b \\ c, d \end{matrix}\,\right|\, 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \quad \text{if } c + d = 1 - n + a + b. \tag{1.3.5}$$

Stirling's formula for the gamma function is

$$\operatorname{Log}\Gamma(z) = \left(z - \frac{1}{2}\right)\operatorname{Log} z - z + \frac{1}{2}\ln(2\pi) + O\left(z^{-1}\right), \tag{1.3.6}$$

$|\arg z| \le \pi - \epsilon$, $\epsilon > 0$. An important consequence of Stirling's formula is

$$\lim_{z\to\infty} z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)} = 1. \tag{1.3.7}$$

The **Pfaff–Kummer transformation** is

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\,\middle|\,z\right) = (1-z)^{-a}{}_2F_1\left(\begin{matrix}a,c-b\\c\end{matrix}\,\middle|\,\frac{z}{z-1}\right), \tag{1.3.8}$$

and is valid for $|z| < 1$, $|z| < |z-1|$. An iterate of (1.3.8) is

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\,\middle|\,z\right) = (1-z)^{c-a-b}{}_2F_1\left(\begin{matrix}c-a,c-b\\c\end{matrix}\,\middle|\,z\right), \tag{1.3.9}$$

for $|z| < 1$. Since ${}_1F_1(a;c;z) = \lim_{b\to\infty} {}_2F_1(a,b;c;z/b)$, (1.3.9) yields

$${}_1F_1(a;c;z) = e^z{}_1F_1(c-a;c;-z). \tag{1.3.10}$$

1.4 q-Series

An analogue of the derivative is the q-difference operator

$$\left(D_q f\right)(x) = \left(D_{q,x} f\right)(x) = \frac{f(x) - f(qx)}{(1-q)x}. \tag{1.4.1}$$

It is clear that

$$D_{q,x} x^n = \frac{1-q^n}{1-q} x^{n-1}, \tag{1.4.2}$$

and for differentiable functions $\lim_{q\to 1^-} (D_q f)(x) = f'(x)$. The **product rule for D_q** is

$$\left(D_q fg\right)(x) = f(x)\left(D_q g\right)(x) + g(qx)\left(D_q f\right)(x). \tag{1.4.3}$$

For finite a and b, $0 < q < 1$ the ***q*-integral** is

$$\int_0^a f(x)\,d_qx := \sum_{n=0}^{\infty}\left[aq^n - aq^{n+1}\right] f\left(aq^n\right), \tag{1.4.4}$$

$$\int_a^b f(x)\,d_qx := \int_0^b f(x)\,d_qx - \int_0^a f(x)\,d_qx. \tag{1.4.5}$$

Moreover,

$$\int_0^\infty f(x)\,d_qx := (1-q)\sum_{n=-\infty}^{\infty} q^n f\left(q^n\right). \tag{1.4.6}$$

The analogue of a change of variable is

$$\int_a^b f(x)g(qx)\,d_qx = q^{-1}\int_a^b g(x)f(x/q)\,d_qx + q^{-1}(1-q)[ag(a)f(a/q) - bg(b)f(b/q)]. \tag{1.4.7}$$

Let

$$x_k := aq^k, \quad y_k := bq^k, \tag{1.4.8}$$

and $w(x_k) > 0$ and $w(y_k) > 0$ for $k = 0, 1, \ldots$. We will take $a \le 0 \le b$ and use the inner product

$$\langle f, g\rangle_q = \int_a^b f(t)\overline{g(t)}w(t)\,d_qt. \tag{1.4.9}$$

A ***q*-analogue of integration by parts** for $D_{q,x}$ is

$$\begin{aligned}\int_a^b D_{q,x}f(t)\overline{g(t)}w(t)\,d_qt &= -f(x_0)\,\overline{g(x_{-1})}w(x_{-1}) + f(y_0)\,\overline{g(y_{-1})}w(y_{-1})\\ &\quad - q^{-1}\left\langle f, \frac{1}{w(x)}D_{q^{-1},x}(g(x)w(x))\right\rangle_q,\end{aligned} \tag{1.4.10}$$

provided that

$$\lim_{n\to\infty} w(x_n)\,f(x_{n+1})\,\overline{g(x_n)} = \lim_{n\to\infty} w(y_n)\,f(y_{n+1})\,\overline{g(y_n)} = 0. \tag{1.4.11}$$

We also let

$$\begin{aligned}\langle f, g\rangle_{q^{-1}} &:= -\frac{(1-q)}{q}\sum_{n=0}^{\infty} f(r_n)\,\overline{g(r_n)}r_n w(r_n)\\ &\quad + \frac{(1-q)}{q}\sum_{n=0}^{\infty} f(s_n)\,\overline{g(s_n)}s_n w(s_n),\end{aligned} \tag{1.4.12}$$

with

$$r_n := \alpha q^{-n}, \quad s_n = \beta q^{-n}, \tag{1.4.13}$$

and w a function positive at r_n and s_n. The analogue of integration by parts is

$$\begin{aligned}\left\langle D_{q^{-1},x}f, g\right\rangle_{q^{-1}} &= -f(r_0)\,\frac{\overline{g(r_{-1})}r_{-1}w(r_{-1})}{r_{-1}-r_0} + f(s_0)\,\frac{\overline{g(s_{-1})}s_{-1}w(s_{-1})}{s_{-1}-s_0}\\ &\quad - q\left\langle f, \frac{x}{w(x)}D_{q,x}(g(x)w(x))\right\rangle_{q^{-1}},\end{aligned} \tag{1.4.14}$$

provided that both sides are well defined and

$$\lim_{n\to\infty}\left[-w(r_n)\,r_n f(r_{n+1})\,\overline{g(r_n)} + w(s_n)\,f(s_{n+1})\,\overline{g(s_n)}\right] = 0. \tag{1.4.15}$$

The ***q*-shifted factorials** are

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=1}^{n}\left(1-aq^{k-1}\right), \quad n=1,2,\ldots, \tag{1.4.16}$$

and if $|q|<1$,

$$(a;q)_\infty := \prod_{k=1}^{\infty}\left(1-aq^{k-1}\right). \tag{1.4.17}$$

The **multiple *q*-shifted factorials** are defined by

$$(a_1,a_2,\ldots,a_k;q)_n := \prod_{j=1}^{k}\left(a_j;q\right)_n. \tag{1.4.18}$$

We shall also use

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{1.4.19}$$

which agrees with (1.4.16) when $\alpha = 0,1,2,\ldots$ but holds for general α when $aq^\alpha \neq q^{-n}$ for a nonnegative integer n. The q-binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{1.4.20}$$

Unless we say otherwise we shall always assume that

$$0<q<1. \tag{1.4.21}$$

A **basic hypergeometric series** (Gasper and Rahman, 2004) is

$$\begin{aligned} {}_r\phi_s\left(\left.\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}\right| q,z\right) &= {}_r\phi_s\left(a_1,\ldots,a_r;b_1,\ldots,b_s;q,z\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n} z^n \left(-q^{(n-1)/2}\right)^{n(s+1-r)}. \end{aligned} \tag{1.4.22}$$

Note that $(q^{-k};q)_n = 0$ for $n = k+1, k+2, \ldots$. If one of the numerator parameters is of the form q^{-k} then the sum on the right-hand side of (1.4.22) is a finite sum and we say that the series in (1.4.22) is **terminating**. A series that does not terminate is called **nonterminating**. The radius of convergence of the series in (1.4.22) is 1, 0, or ∞ for $r = s+1$, $r > s+1$, and $r < s+1$, respectively, as can be seen from the ratio test.

A basic hypergeometric series is called **balanced** if $r = s+1$ and $q\prod_{j=1}^{s+1} a_j = \prod_{j=1}^{s} b_j$. The Sears transformation connects two balanced terminating ${}_4\phi_3$. It is (Gasper and Rahman, 2004, (III.15))

$${}_4\phi_3\left(\left.\begin{matrix} q^{-n},a,b,c \\ d,e,f \end{matrix}\right| q,q\right) = \left(\frac{bc}{d}\right)^n \frac{(de/bc, df/bc;q)_n}{(e,f;q)_n}\, {}_4\phi_3\left(\left.\begin{matrix} q^{-n},a,d/b,d/c \\ d,de/bc,df/bc \end{matrix}\right| q,q\right), \tag{1.4.23}$$