Cambridge University Press
978-0-521-19573-7 - Atmospheric Dynamics
Mankin Mak
Excerpt
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1 Fundamental concepts and physical laws

We begin by discussing the fundamental concepts and the laws of physics that underlie the formulation of any dynamical analysis of atmospheric problems. We elaborate on how to conceptualize a fluid medium and how to mathematically quantify the state of a fluid in Section 1.1. We proceed to review a set of physical laws that govern the behavior of all disturbances in a dry atmosphere setting, detailing their physical meaning and mathematical representations in Sections 1.2 through 1.6. The observed distributions of two important mean characteristics of the atmosphere, namely its stratification and baroclinicity, are finally examined in Section 1.7.

1.1 Basic notions

1.1.1 Continuum hypothesis

Although the atmosphere may be visualized as a collection of randomly moving gas molecules in constant collision with one another, we conceptualize it as a fluid when we make dynamical atmospheric analyses by invoking a *continuum hypothesis*. We only seek to quantify the *macroscopic properties* of the atmosphere. One such property is the wind defined to be the average velocity of the molecules in a very small volume about each location. We call such a loosely defined amount of air molecules an atmospheric fluid parcel. We therefore think of the atmosphere as a *fluid medium* consisting of innumerable *parcels*. Furthermore, we assume that each property of the atmosphere varies continuously in time and space, whereby we may evaluate their derivatives with respect to the space variables and time in an analysis.

1.1.2 Lagrangian vs. Eulerian descriptions of a fluid

The fluid in a system can be described in principle in terms of the position of every one of its innumerably large number of fluid elements. The position may be measured by its Cartesian coordinates $(x(t), y(t), z(t))$ and its velocity components by $u(t) = dx/dt, v(t) = dy/dt, w(t) = dz/dt$ at all times. This is known as a *Lagrangian description* of a fluid with time being the only independent variable. A fluid element would deform indefinitely when it is subject to non-uniform stresses. Its shape and size would change continually making it impossible for us to track its identity for a long time. Hence, it is not feasible to perform a Lagrangian analysis of a fluid except for a short duration. A much more convenient means of describing a fluid is to quantify all of its fundamental properties at a fixed array of points in the domain at all times. We do not concern ourselves with the identity of the individual fluid parcels that happen to be at various points and time. This is known as an *Eulerian description* of a fluid. One fundamental property is the velocity field, $\vec{V} \equiv (u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$, where the independent variables are x, y, z and t. We will talk about the other fundamental properties shortly.

In a Lagrangian description the rate of change of any property of a fluid parcel, $\xi(t)$, in time as it moves about is called a *total derivative*, $D\xi/Dt$. In an Eulerian description, the rate of change of any property $\xi(x,y,z,t)$ at a fixed point is called a *local derivative*, $\partial\xi/\partial t$. The relationship between these two derivatives is established as follows. Suppose a generic property of an air parcel in a Lagrangian sense, $\xi(x(t), y(t), z(t), t)$, changes by $d\xi$ when its position has changed by (dx, dy, dz) in a time interval dt. They are interrelated by the chain rule of calculus

$$d\xi = \frac{\partial\xi}{\partial x}dx + \frac{\partial\xi}{\partial y}dy + \frac{\partial\xi}{\partial z}dz + \frac{\partial\xi}{\partial t}dt. \tag{1.1}$$

The partial derivatives in (1.1) are however in the Eulerian sense. Each of them is to be determined with the other three of the four independent variables (x, y, z, t) fixed. Upon dividing (1.1) through by dt, we get

$$\frac{D\xi}{Dt} \equiv \frac{d\xi}{dt} = \frac{\partial\xi}{\partial x}\frac{dx}{dt} + \frac{\partial\xi}{\partial y}\frac{dy}{dt} + \frac{\partial\xi}{\partial z}\frac{dz}{dt} + \frac{\partial\xi}{\partial t}. \tag{1.2}$$

This equation relates a Lagrangian description to an Eulerian description of a fluid. A more compact notation for (1.2) is

$$\frac{D\xi}{Dt} = \frac{\partial\xi}{\partial t} + \vec{V}\cdot\nabla\xi, \tag{1.3}$$

where $\vec{V} = (u, v, w) \equiv \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$, $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the grad operator in Cartesian coordinates. Equation (1.3) can be of course applied in any coordinate system as long as $\vec{V}$ and ∇ are consistently defined. It is instructive to rewrite (1.3) as

$$\frac{\partial\xi}{\partial t} = \frac{D\xi}{Dt} - \vec{V}\cdot\nabla\xi. \tag{1.4}$$

Equation (1.4) says that:

Local rate of change of ξ at a point
$=$ Rate of change of ξ in a parcel caused by external processes $\left(\frac{D\xi}{Dt}\right)$
$+$ Rate of change of ξ due to the advective process at a point, $(-\vec{V}\cdot\nabla\xi)$.

The advective rate of change of ξ at a point is positive when the velocity at that point is directed from a region of higher values of ξ towards a region of lower values of ξ. One part of it $(-u\xi_x - v\xi_y)$ is called horizontal advection of ξ and the other part $(-w\xi_z)$ is called vertical advection. The horizontal velocity components of air parcels in a large atmospheric disturbance are much larger than the vertical component. Those air parcels move quasi-horizontally. What we can reliably measure with a network of weather stations is only this component of airflow. It is what we usually call the *wind*,

$\vec{V}_2 \equiv (u, v)$. The advective rate of change in that case is mostly associated with the horizontal advection. If ξ stands for zonal velocity, u, then $-\vec{V}\cdot\nabla\xi \approx -\vec{V}_2\cdot\nabla u$ would be advection of zonal momentum. If ξ stands for temperature, T, then $-\vec{V}\cdot\nabla\xi \approx -\vec{V}_2\cdot\nabla T$ would be thermal advection.

1.1.3 Physical laws

Whatever happens in the atmosphere must be compatible with the laws of physics. The pertinent laws are listed below for quick reference:

- *Conservation of Mass* asserts that the mass of a fluid element remains unchanged no matter how it moves and how all its properties change.
- *Conservation of Momentum* is the underlying principle of Isaac Newton's three laws of mechanics. The latter are concerned with how the momentum of an object changes under the influence of forces.
- *Law of Gravitation* is a statement of the force acting on an object due to the presence of another object.
- The forces associated with the momentum exchange between a fluid parcel in contact with another fluid parcel are described by the notions of pressure and viscosity.
- *Equation of State* is the identity signature of the fluid medium under consideration.
- *Conservation of Energy* is the basis of the *First Law of Thermodynamics*. It quantifies how one might evaluate the rate of change of temperature of a fluid element in different circumstances.
- The laws of radiative transfer and phase transition of water in the atmosphere are approximately incorporated in an indirect manner when we analyze the basic dynamics of an atmospheric flow if necessary.

The mathematical expressions for these laws of physics take on the form of a complete set of partial differential equations. The solutions of those equations are very general for they could describe all classes of disturbances. The different concepts and physical laws are elaborated in the following sections of this chapter.

1.2 Laws of mechanics

The location and movement of an object are measured in the context of a reference frame. A special reference frame is an *inertial reference frame*, which is defined as one that moves at a constant speed in a certain direction. It follows that there is no unique inertial reference frame. A special one is absolutely at rest in space at all times. Such a reference frame presupposes that the notions of absolute space and absolute time are meaningful.

Newton formulated three laws of mechanics in conjunction with the use of an inertial reference frame. They are:

(1st Law) An object in uniform motion continues to be in the same uniform motion when it is not under the influence of a net force.

(2nd Law) When there is a force, $\vec{F}$, acting upon an object, the latter accelerates at a rate proportional to the force and inversely proportional to its mass, m,

$$\frac{d\vec{v}}{dt} = \frac{\vec{F}}{m} \tag{1.5}$$

where $\vec{v}$ is velocity in an inertial reference frame.

(3rd Law) If two bodies exert forces upon each other, these forces are equal in magnitude and opposite in direction. This law is applicable only if the force exerted by one object on another object is directed along the line connecting the two objects. Two known forces are of this type: gravitational force between two objects and electrostatic force between two electric charges. This law is not applicable for a case in which a force is velocity dependent.

Newton's three laws are in essence three statements concerned with the absolute momentum of an object defined as

$$\vec{P} = m\vec{v} \tag{1.6}$$

under three different circumstances. The 1st Law effectively says that the absolute momentum of an object conserves in the absence of external force. The 2nd Law effectively says that absolute momentum of an object under the influence of a force $\vec{F}$ changes in time at a rate equal to

$$\frac{d\vec{P}}{dt} = \vec{F}. \tag{1.7}$$

Suppose two objects with absolute momentum $\vec{P}_j$, $j = 1, 2$ interact with one another. According to Newton's 3rd Law, we have $\vec{F}_1 = -\vec{F}_2$ where the force acting on body-1 due to body-2 is $\vec{F}_1$ and vice versa is $\vec{F}_2$. Then by using (1.7) we have

$$\frac{d\left(\vec{P}_1 + \vec{P}_2\right)}{dt} = 0 \tag{1.8}$$

which simply means that the total momentum of a composite system would not change as a consequence of interactions among its members.

In passing, it should be emphasized that the concepts of absolute space and absolute time are ad hoc notions. Their logical implications are incompatible with the observation that the velocity of light is the same regardless of the movement of the light source. Einstein's special theory of relativity adopts that as a postulate and supersedes Newton's laws of mechanics since it accurately governs the movement of objects even in a speed close to the velocity of light. That theory establishes that the notions of absolute space and absolute time are superfluous and that space and time are intrinsically related depending on the motion of the observer. Newton's theory is nevertheless sufficiently accurate for atmospheric studies since wind speed is negligibly slow compared to the velocity of light.

1.2.1 Inertial vs. non-inertial reference frames

Wind is the movement of air parcels measured relative to a weather station fixed on Earth. The reference frame for measuring wind is therefore a non-inertial reference frame because

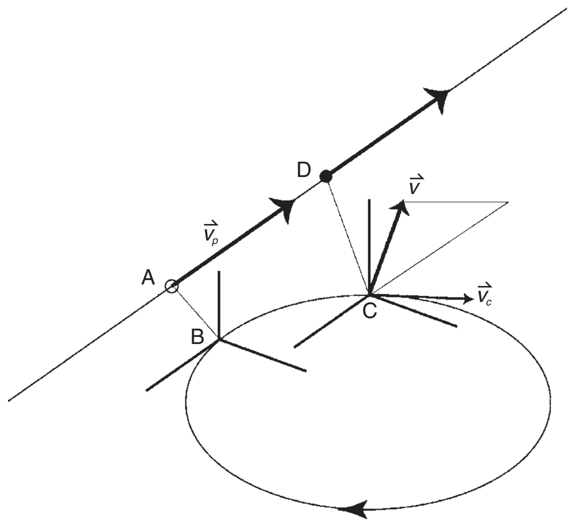

Fig. 1.1 Depiction of a particle moving in constant velocity along AD in an inertial reference frame and in a non-inertial reference frame (another observer moving in an elliptical orbit). The particle's velocity is zero to the second observer at B and is indicated by an arrow to this observer at C.

the Earth rotates about its own axis and also moves around the Sun. The total movement of each air parcel as viewed from space is quite complicated indeed. Let us compare the description of the movement of an object in the absence of any force $\left(\vec{F}=0\right)$ in an inertial reference frame with the description in a non-inertial reference frame. The situation is depicted in Fig. 1.1. Suppose this object moves at a constant velocity $\vec{v}_p$ along a straight line with respect to an observer in an inertial reference frame. Its movement to an observer in an inertial reference frame is clearly very different to another observer in a reference frame that moves in an elliptical orbit with a constant speed $|\vec{v}_c| = \left|\vec{v}_p\right|$. Suppose the observer is at point B when the object is at point A and would have moved to C when the object moves to D. The object is momentarily stationary to observer at B as far as he can tell, $\left(\vec{v}=\vec{v}_p-\vec{v}_c=0\right)$. Later at location C, the object appears to be moving in the direction indicated by the arrow, $\left(\vec{v}=\vec{v}_p-\vec{v}_c\neq 0\right)$. Hence, this observer concludes that the object continually accelerates as it moves. It follows that the equation of motion to him is NOT $\vec{F}=0=\dfrac{d}{dt}(m\vec{v})$ where $\vec{v}$ is the velocity with respect to him as the observer. He would conclude that there must be a force acting on the object if he wishes to use the form of Newton's formula (1.5) for analyzing its motion.

1.3 Equations of motion in a rotating reference frame

Since the Earth rotates about its own axis and moves around the Sun, it would be most convenient to use a non-inertial reference frame fixed on Earth when we analyze atmospheric disturbances. This reference frame rotates once a day. The movement of air parcels

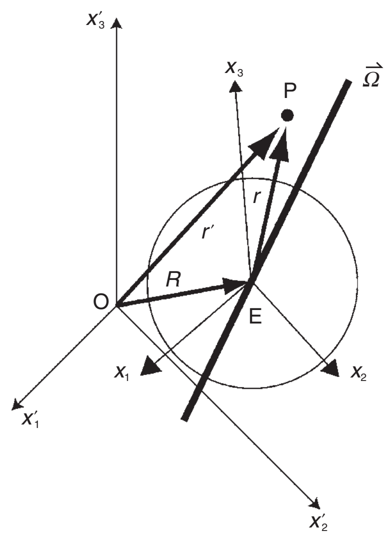

Fig. 1.2 The x_i' are coordinates of an inertial reference frame; x_i are coordinates of a rotating reference frame. E is center of such reference frame; $\vec{\Omega}$ is the vector representation of the rotation of the reference frame; $\vec{R}$ indicates the position of the origin of the non-inertial frame; $\vec{r}$ indicates the position of an object P in the non-inertial frame; $\vec{r}'$ indicates the position of the object in the inertial frame.

is measured relative to the surface of the Earth. We will derive in the following subsection an equation that adequately governs the motion of an air parcel in this framework. It is written in a vector form applicable to any choice of coordinate system. It takes on the form of $(d\vec{v}/dt)_{rotating} = \vec{F}/m - \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) - 2\vec{\Omega} \times \vec{v}$ where $\vec{\Omega}$ is the vector representing the Earth's rotation about its own axis, $\vec{v}$ is the velocity of the air parcel in this rotating frame, $\vec{r}$ is the position vector of the air parcel from the center of the Earth, m the mass of the air parcel and $\vec{F}$ the net force acting on the air parcel.

1.3.1 General derivation

By definition, the real forces acting on an air parcel are the same regardless of the particular reference coordinates adopted in an analysis. So, we only need to transform the expression of acceleration from an inertial reference frame to a non-inertial reference frame. Without loss of generality, we use Cartesian coordinates to depict an object in both an inertial frame and a rotating reference frame as indicated in Fig. 1.2. Let us focus on an air parcel with mass m at point P. The x_i' $(i = 1,2,3)$ are the coordinates of an inertial reference frame according to the *right-hand convention*. The origin of this inertial reference frame is arbitrary. The x_i are the coordinates of a non-inertial reference frame whose origin is at the center of the Earth. $\vec{R}$ measures the position of the center of the Earth relative to the origin of the inertial frame. This non-inertial reference frame is completely general in that it may move and accelerate, $\dot{\vec{R}} \equiv \dfrac{d\vec{R}}{dt} \neq 0$, $\ddot{\vec{R}} \neq 0$. It also may rotate about an arbitrary axis in a varying rate, $\dot{\vec{\Omega}} \neq 0$.

The following notations are introduced for the derivation:
$\vec{R}$ = position vector for the origin of the rotating system in an inertial (fixed) frame
$\vec{e}_i$ = unit vectors of the rotating reference frame
$\vec{r} = \sum_{i=1}^{3} x_i \vec{e}_i$ = position vector of P in the rotating system
$\vec{r}\,' = \vec{R} + \vec{r}$ = position vector of P in a fixed system
$\vec{\Omega} = \sum_i \omega_i \vec{e}_i$ = rotation vector of the rotating system.

Then we have

$$\left(\frac{d\vec{r}\,'}{dt}\right) = \left(\frac{d\vec{R}}{dt}\right) + \frac{d}{dt}\sum_i (x_i \vec{e}_i) = \left(\frac{d\vec{R}}{dt}\right) + \sum_i \dot{x}_i\,\vec{e}_i + \sum_i x_i\,\dot{\vec{e}}_i. \tag{1.9}$$

The relative velocity is $(d\vec{r}/dt)_{rotating} = \sum_i \dot{x}_i \vec{e}_i$. We make use of two simple properties of vectors to determine $\dot{\vec{e}}_i$,

$$\vec{D} = \left(\vec{E} + \vec{F}\right) \times \vec{G} = \vec{E} \times \vec{G} + \vec{F} \times \vec{G},$$

$$\vec{C} = \vec{A} \times \vec{B} = (A_2 B_3 - A_3 B_2)\vec{e}_1 + (A_3 B_1 - A_1 B_3)\vec{e}_2 + (A_1 B_2 - A_2 B_1)\vec{e}_3.$$

Vector $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$ as evidenced by a special case $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$. It follows that the change of $\vec{e}_1$ in time arises from two components of $\vec{\Omega}$: ω_2 and ω_3. The change of $\vec{e}_1$ in time δt arising from ω_3 about the x_3 axis (pointing out of the paper) is $\delta\vec{e}_1 = \omega_3 \delta t \vec{e}_2$. Likewise, the change of $\vec{e}_1$ in time δt arising from ω_2 about the x_2 axis (pointing into the paper) is $\delta\vec{e}_1 = -\omega_2 \delta t \vec{e}_3$. Thus the total rate of change of $\vec{e}_1$ is

$$\dot{\vec{e}}_1 \equiv \frac{d\vec{e}_1}{dt} = \omega_3 \vec{e}_2 - \omega_2 \vec{e}_3 = \vec{\Omega} \times \vec{e}_1 = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ 1 & 0 & 0 \end{vmatrix}. \tag{1.10}$$

Similarly $\dfrac{d\vec{e}_2}{dt} = -\omega_3 \vec{e}_1 + \omega_1 \vec{e}_3$ $\dfrac{d\vec{e}_3}{dt} = \omega_2 \vec{e}_1 - \omega_1 \vec{e}_2$.

Thus we have in general vector notation

$$\dot{\vec{e}}_i = \vec{\Omega} \times \vec{e}_i, \qquad i = 1, 2, 3.$$

By (1.9), (1.10)

$$\left(\frac{d\vec{r}\,'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \left(\frac{d\vec{r}}{dt}\right)_{rotating} + \vec{\Omega} \times \vec{r}. \tag{1.11}$$

Equation (1.11) written in a more concise notation is $\vec{v}\,' = \dot{\vec{R}} + \vec{v} + \vec{\Omega} \times \vec{r}$. Taking the time derivative of (1.11) in the context of the inertial system, we get

$$\begin{aligned} \left(\frac{d\vec{v}\,'}{dt}\right)_{fixed} &= \ddot{\vec{R}} + \left(\frac{d\vec{v}}{dt}\right)_{fixed} + \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\frac{d\vec{r}}{dt}\right)_{fixed} \\ &= \ddot{\vec{R}} + \left(\frac{d\vec{v}}{dt}\right)_{rotating} + \vec{\Omega} \times \vec{v} + \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \vec{v} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right). \end{aligned} \tag{1.12}$$

According to Newton's law, the equation of motion for P in an inertial reference frame is simply $\left(\frac{d\vec{v}'}{dt}\right)_{fixed} = \frac{\vec{F}}{m}$ where $\vec{F}$ is the net real force. With the use of (1.12), we can then write Newton's equation of motion in a general rotating reference frame as

$$\underset{\text{(i)}}{\frac{\vec{F}}{m}} - \underset{\text{(ii)}}{\ddot{\vec{R}}} - \underset{\text{(iii)}}{\dot{\vec{\Omega}} \times \vec{r}} - \underset{\text{(iv)}}{\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)} - \underset{\text{(v)}}{2\vec{\Omega} \times \vec{v}} \equiv \frac{\vec{F}_{eff}}{m} = \left(\frac{d\vec{v}}{dt}\right)_{rotating.} \tag{1.13}$$

Equation (1.13) is written in the same format as that for an inertial reference frame. It should be emphasized that although Cartesian coordinates are used in the derivation above, we are now writing the final equation in vector notation. It would need to be written explicitly for a specific coordinate system in a particular application.

The five terms on the LHS of (1.13) are to be interpreted as "forces" acting on an air parcel. Since only term (i) represents the real forces by definition, the others are *apparent forces*, which stem solely from the use of the non-inertial reference frame under consideration. Their sum constitutes an effective net force that would give rise to a relative acceleration.

Term (ii) is associated with the sum of the Earth's movement around the Sun, the Sun's movement around the Milky Way, etc. A part of (i) must be responsible for (ii). That is essentially the net gravitational force from all celestial bodies exerted on the Earth. It must necessarily cancel (ii). If we neglect (ii) we must not include the gravitational force due to all celestial bodies except the Earth itself in (i) for consistency.

Term (iii) is proportional to $\dot{\Omega}$ associated with the precession of the axis of rotation of the Earth. The orientation of Earth's rotational axis changes in a cycle of approximately 26 000 years, like a wobbling top. This term is therefore very small compared to (i). There are additional details concerning the precession of the Earth's axis. We will neglect the precession of the rotation of the Earth altogether in atmospheric applications and thereby assume $\dot{\Omega} = 0$.

Term (iv) is the *centrifugal force* on the air parcel associated with the rotation of the non-inertial reference frame. It is in the direction outward normal to the axis of rotation.

Term (v) is called *Coriolis force* associated with the movement of the air parcel at P as measured in the rotating reference frame. Its direction is perpendicular to both the rotation vector and the relative velocity. Its magnitude is proportional to the speed of the object $|\vec{v}|$ and to twice the rotation rate of the Earth, 2Ω.

Summing up, we may simplify (1.13) to

$$\frac{\vec{F}}{m} - \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) - 2\vec{\Omega} \times \vec{v} = \left(\frac{d\vec{v}}{dt}\right)_{rotating} \tag{1.14}$$

as an adequately accurate equation of motion for any air parcel in atmospheric studies.

1.3.2 Physical nature of the Coriolis force

Although the mathematical derivation of the expression for the Coriolis force in (1.14) is rigorous, it does not give us a feel for this force from a physical point of view. To develop a better feel, we re-derive the form of Coriolis force with more elemental but physical

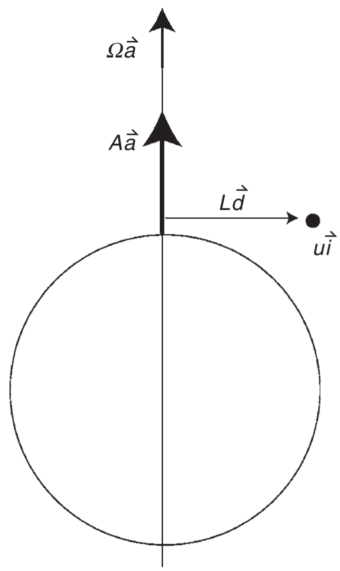

Fig. 1.3 Schematic of an air parcel on a meridional plane in the atmosphere. The Earth's rotation vector, position vector, relative zonal velocity and absolute angular momentum of the parcel are denoted by $\Omega\vec{a}$, $L\vec{d}$, $u\vec{i}$ and $A\vec{a}$ respectively; $\vec{i}$ is a unit vector pointing into the plane.

reasoning in two scenarios. In the first scenario, we consider an air parcel of unit mass initially at rest at a latitude φ. Its location is at a distance L from the Earth's axis of rotation. If its absolute motion synchronizes with the Earth's rotation, there would be no relative motion and there would be a radially inward acceleration (known as centripetal acceleration). A change of its sign can be reinterpreted as a force in a radially outward direction acting on the air parcel (known as *centrifugal force*, $\vec{F}_{cen}$). This is the force required to sustain it at this location in a rotating coordinate. For convenience, we will make use of two different coordinate systems in the following discussion. We will consider two scenarios in which the air parcel is set to simple motion. We first describe what happens in a *polar-cylindrical coordinate system* $(\vec{i}, \vec{d}, \vec{a})$ where $\vec{i}$ is a unit vector pointing eastward, $\vec{d}$ a unit vector pointing away from the Earth's axis of rotation, and $\vec{a}$ a unit vector directed along the rotation axis. The rotation vector of the Earth is then $\vec{\Omega} = \Omega\vec{a}$. The initial position vector of the air parcel can be designated by $\vec{L} = L\vec{d}$. Its absolute velocity is $\vec{V} = \vec{\Omega} \times \vec{L} = \Omega L\vec{i}$ in the azimuthal direction and its centripetal acceleration is $\vec{\Omega} \times \vec{V} = -\Omega^2 L\vec{d}$ pointing towards the axis of rotation. The centrifugal force is then $\vec{F}_{cen} = \Omega^2 L\vec{d}$. The mathematical expression for the *angular momentum*, $\vec{A}$, of this air parcel is

$$\vec{A} = \vec{L} \times \vec{V} = \Omega L^2 \vec{a}, \tag{1.15}$$

where $\vec{A}$ is a vector in the direction of the rotation axis with a magnitude equal to the product of the tangential velocity about the axis of rotation and the distance from the axis (moment of momentum). According to Newton's law, the angular momentum $\vec{A}$ of the air parcel would not change in time if it is not acted upon by an external torque. The configuration is depicted in Fig. 1.3.

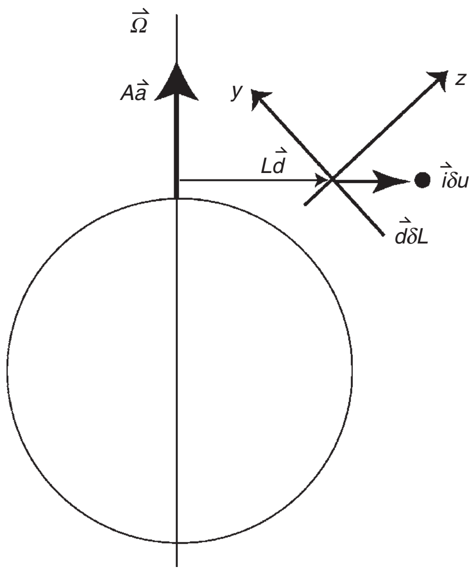

Fig. 1.4 Schematic of a displaced air parcel on a meridional plane in the atmosphere.

Suppose the air parcel is somehow displaced infinitesimally outward to a distance $L + \delta L$ in a short time interval δt. We want to deduce what might be the consequence of this displacement. Since no torque is required for this displacement, its absolute angular momentum would not change. This implies a change in its azimuthal (zonal) velocity, δu, since its distance from the axis has changed:

$$\begin{aligned}(A\vec{a})_{initial} &= (A\vec{a})_{later} \\ \Omega L^2\vec{a} &= (L+\delta L)\vec{d} \times (\Omega(L+\delta L)+\delta u)\vec{i} \\ &= \left(\Omega L^2 + 2\Omega L\,\delta L + L\,\delta u + \delta L\,\delta u\right)\vec{a}.\end{aligned} \tag{1.16}$$

For an infinitesimal displacement, the product term $\delta L\delta u$ would be negligibly smaller than other terms. Then it follows

$$\delta u = -2\Omega\,\delta L. \tag{1.17}$$

Equation (1.17) says that if an air parcel is displaced outward from the axis of rotation, it would necessarily move to the west, $\delta u < 0$. This movement would be observable as a zonal velocity in a rotating reference frame. This relation is analogous to an ice skater who could slow down his spinning motion by stretching his arms outward.

Now let us represent this radial displacement in a local Cartesian coordinate system with its origin collocated at the air parcel. $\left(\vec{i},\vec{j},\vec{k}\right)$ are the unit vectors. The result is $\delta L\vec{d} = \delta y\vec{j} + \delta z\vec{k}$ (Fig. 1.4; δy would have a negative value in this case). The contribution to (1.17) from δL comes from two parts: $\delta L_1 = -\delta y \sin\varphi$ and $\delta L_2 = \delta z\cos\varphi$ where φ is the latitude. The contributions to δu induced by the two parts of δL are additive. Thus, (1.17) may be written as

$$\delta u = 2\Omega\sin\varphi\,\delta y - 2\Omega\cos\varphi\,\delta z. \tag{1.18}$$