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978-0-521-19503-4 - Stochastic Control and Mathematical Modeling: Applications in Economics

Hiroaki Morimoto

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Stochastic Control and Mathematical Modeling

Applications in Economics

This is a concise and elementary introduction to stochastic control and mathematical modeling. This book is designed for researchers in stochastic control theory studying its application in mathematical economics and those in economics who are interested in mathematical theory in control. It is also a good guide for graduate students studying applied mathematics, mathematical economics, and nonlinear PDE theory.

Contents include the basics of analysis and probability, the theory of stochastic differential equations, variational problems, problems in optimal consumption and in optimal stopping, optimal pollution control, and solving the Hamilton–Jacobi–Bellman equations with boundary conditions. Major mathematical requisitions are contained in the preliminary chapters or in the appendix so that readers can proceed without referring to other materials.

Hiroaki Morimoto is a Professor in Mathematics at the Graduate School of Science and Engineering at Ehime University. His research interests include stochastic control, mathematical economics and finance and insurance applications, and the viscosity solution theory.

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HIROAKI MORIMOTO

Ehime University, Matsuyama, Japan



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To My Teacher M. Nisio

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Hiroaki Morimoto

Frontmatter

[More information](#)

Contents

Preface *page xi*

Part I Stochastic Calculus and Optimal Control Theory

1	Foundations of Stochastic Calculus	3
1.1	Review of Probability	3
1.2	Stochastic Processes	9
1.3	Stopping Times	12
1.4	Martingales	13
1.5	Stochastic Integrals	17
1.6	Itô's Formula	27
1.7	Stochastic Differential Equations: Strong Formulation	33
1.8	Martingale Moment Inequalities	37
1.9	Existence and Uniqueness: Locally Lipschitz Case	40
1.10	Comparison Results	44
1.11	The Martingale Representation Theorem	47
2	Stochastic Differential Equations: Weak Formulation	51
2.1	Probability Laws	51
2.2	Linear Functionals and Probabilities	56
2.3	Regular Conditional Probabilities	60
2.4	Weak Solutions	64
2.5	Uniqueness in Law	69
2.6	Markov Properties	73
3	Dynamic Programming	77
3.1	Dynamic Programming Principle: Deterministic Case	78
3.2	Dynamic Programming Principle: Stochastic Case	81
3.3	Dynamic Programming Principle: Polynomial Growth	90
3.4	The HJB Equations: Stochastic Case	93

4	Viscosity Solutions of Hamilton–Jacobi–Bellman Equations	97
4.1	Definition of Viscosity Solutions	97
4.2	The HJB Equations: First-Order Case	99
4.3	The HJB Equations: Second-Order Case	103
4.4	Uniqueness of Viscosity Solutions	107
4.5	Stability	121
4.6	Viscosity Solutions and Markov Processes	124
5	Classical Solutions of Hamilton–Jacobi–Bellman Equations	128
5.1	Linear Elliptic Equations: Weak Solutions	129
5.2	Linear Elliptic Equations: Classical Solutions	146
5.3	The Dirichlet Problem for HJB Equations	151
5.4	Stochastic LQ Problems with Constraints	162
Part II Applications to Mathematical Models in Economics		
6	Production Planning and Inventory	171
6.1	The Model	171
6.2	Viscosity Solutions of the HJB Equations	172
6.3	Classical Solutions	176
6.4	Optimal Production Planning	178
7	Optimal Consumption/Investment Models	185
7.1	The Model	185
7.2	HARA Utility	187
7.3	HJB Equations	189
7.4	Optimal Policies	193
8	Optimal Exploitation of Renewable Resources	197
8.1	The Model	197
8.2	Viscosity Solutions of the HJB Equations	201
8.3	Concavity and Regularity	207
8.4	Optimal Exploitation	211
8.5	Examples	215
9	Optimal Consumption Models in Economic Growth	217
9.1	The Model	217
9.2	HJB Equations	219
9.3	Viscosity Solutions	220
9.4	Classical Solutions	233
9.5	Optimal Policies	233
10	Optimal Pollution Control with Long-Run Average Criteria	237
10.1	The Model	237
10.2	Moments	239
10.3	The HJB Equations with Discount Rates	240

Cambridge University Press

978-0-521-19503-4 - Stochastic Control and Mathematical Modeling: Applications in Economics

Hiroaki Morimoto

Frontmatter

[More information](#)

Contents	ix
10.4 Solution of the HJB Equation	247
10.5 Optimal Policies	249
11 Optimal Stopping Problems	252
11.1 The Model	252
11.2 Remarks on Variational Inequalities	253
11.3 Penalized Problem	254
11.4 Passage to the Limit as $\varepsilon \rightarrow 0$	258
11.5 Viscosity Solutions of Variational Inequalities	262
11.6 Solution of the Optimal Stopping Problem	266
12 Investment and Exit Decisions	269
12.1 The Model	269
12.2 Penalized Problem	271
12.3 Nonlinear Variational Inequalities	280
12.4 Optimal Policies	285
 Part III Appendices	
A Dini's Theorem	291
B The Stone–Weierstrass Theorem	292
C The Riesz Representation Theorem	294
D Rademacher's Theorem	297
E Vitali's Covering Theorem	299
F The Area Formula	301
G The Brouwer Fixed-Point Theorem	308
H The Ascoli–Arzelà Theorem	314
Bibliography	317
Index	323

Cambridge University Press

978-0-521-19503-4 - Stochastic Control and Mathematical Modeling: Applications in Economics

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Frontmatter

[More information](#)

Preface

The purpose of this book is to provide a fundamental description of stochastic control theory and its applications to dynamic optimization in economics. Its content is suitable particularly for graduate students and scientists in applied mathematics, economics, and engineering fields.

A stochastic control problem poses the question: what is the optimal magnitude of a choice variable at each time in a dynamical system under uncertainty? In stochastic control theory, the state variables and control variables, respectively, describe the random phenomena of dynamics and inputs. The state variable in the problem evolves according to stochastic differential equations (SDE) with control variables. By steering of such control variables, we aim to optimize some performance criteria as expressed by the objective functional. Stochastic control can be viewed as a problem of decision making in maximization or minimization. This subject has created a great deal of mathematics as well as a large variety of applications in economics, mathematical finance, and engineering.

This book provides the basic elements of stochastic differential equations and stochastic control theory in a simple and self-contained way. In particular, a key to the stochastic control problem is the dynamic programming principle (DPP), which leads to the notion of viscosity solutions of Hamilton–Jacobi–Bellman (HJB) equations. The study of viscosity solutions, originated by M. Crandall and P. L. Lions in the 1980s, provides a useful tool for dealing with the lack of smoothness of the value functions in stochastic control. The main idea used to solve this maximization problem is summarized as follows:

- (a) We formulate the problem and define the supremum of the objective functional over the class of all control variables, which is called the *value function*.
- (b) We verify that the DPP holds for the value function.
- (c) By the DPP, the value function can be viewed as a unique viscosity solution of the HJB equation associated with this problem.
- (d) The uniform ellipticity and the uniqueness of viscosity solutions show the existence of a unique classical solution to the boundary value problem of

the HJB equation. This gives the smoothness of the viscosity solution of the HJB equation.

- (e) We seek a candidate of optimal control by using the HJB equation. By using Itô's formula, we show the optimality.

This book is divided into three parts: Part I – Stochastic Calculus and Optimal Control Theory; Part II – Applications to Mathematical Models in Economics; and Part III – a collection of appendices providing background materials.

Part I consists of Chapters 1–5. In Chapter 1, we present the elements of stochastic calculus and SDEs, and in Chapter 2, we present the formulation of the weak solutions of SDEs, the concept of regular conditional probability, the Yamada–Watanabe theorem on weak and strong solutions, and the Markov property of a solution of SDE.

In Chapter 3, we introduce the DPP to issue (b). The verification of the DPP is rather difficult compared to the deterministic case. The Yamada–Watanabe theorem in Chapter 2 makes its proof exact. The supremum of (a) is taken over all systems in the weak sense.

Chapter 4 provides the theory of viscosity solutions of the HJB equations for (c). Using Ishii's lemma, we show the uniqueness results on viscosity solutions.

Chapter 5 is devoted to the boundary value problem of the HJB equations for (d) in the classical sense. Section 5.4 explains how to apply (a)–(e) in stochastic control.

Part II consists of Chapters 6–12. Here we present diverse applications of stochastic control theory to the mathematical models in economics. In Chapters 6–10, we take the state variables in these models as the remaining stock of a resource, the labor supply, and the price of the stock. The criteria in the maximization procedure are often given by the utility function of consumption rates as the control variables. Along (a)–(e), an optimal control is shown to exist.

Chapters 11 and 12 deal with the linear and nonlinear variational inequalities, instead of the HJB equations, which are associated with the stopping time problem. The variational inequality is analyzed by the viscosity solutions approach for optimality.

Part III consists of Appendices A–H. These provide some background material for understanding stochastic control theory as quickly as possible.

The prerequisites for this book are basic probability theory and functional analysis (see e.g., R. B. Ash [2], H. L. Royden [139], and A. Friedman [69]). See M. I. Kamien and N. L. Schwartz [80], A. C. Chiang [33], A. K. Dixit and R. S. Pindyck [46], L. Ljungqvist and T. J. Sargent [107], and R. S. Merton [114], for economics references.

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978-0-521-19503-4 - Stochastic Control and Mathematical Modeling: Applications in Economics

Hiroaki Morimoto

Frontmatter

[More information](#)

Preface

xiii

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