Part I

Stochastic Calculus and Optimal Control Theory

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Foundations of Stochastic Calculus

We are concerned here with a stochastic differential equation,

$$
\begin{aligned}
dX(t) &= b(X(t))dt + \sigma(X(t))dB(t), \quad t \geq 0, \\
X(0) &= x \in \mathbf{R}^N,
\end{aligned}
$$

in N-dimensional Euclidean space $\mathbf{R}^N$. Here b, σ are Lipschitz functions, called the *drift term* and the *diffusion term*, respectively, and $\{B(t)\}$ is a standard Brownian motion equation defined on a probability space $(\Omega, \mathcal{F}, P)$. This equation describes the evolution of a finite-dimensional dynamical system perturbed by noise, which is formally given by $dB(t)/dt$. In economic applications, the stochastic process $\{X(t)\}$ is interpreted as the labor supply, the price of stocks, or the price of capital at time $t \geq 0$. We present a reasonable definition of the second term with uncertainty and basic elements of calculus on the stochastic differential equation, called *stochastic calculus*.

A. Bensoussan [16], I. Karatzas and S. E. Shreve [87], N. Ikeda and S. Watanabe [75], I. Gihman and A. Skorohod [72], A. Friedman [68], B. Øksendal [132], D. Revuz and M. Yor [134], R. S. Liptzser and A. N. Shiryayev [106] are basic references for this chapter.

1.1 Review of Probability

1.1.1 Random Variables

Definition 1.1.1. *A triple $(\Omega, \mathcal{F}, P)$ is a* probability space *if the following assertions hold:*

(a) *Ω is a set.*
(b) *$\mathcal{F}$ is a σ-algebra, that is, $\mathcal{F}$ is a collection of subsets of Ω such that*
  (i) *$\Omega, \ \phi \in \mathcal{F}$,*
  (ii) *if $A \in \mathcal{F}$, then $A^c := \Omega \setminus A \in \mathcal{F}$,*
  (iii) *if $A_n \in \mathcal{F}, n = 1, 2, \ldots,$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.*

(c) *P is a probability measure, that is, a map $P : \mathcal{F} \to [0, 1]$, such that*
   (i) $P(\Omega) = 1$,
   (ii) *if $A_n \in \mathcal{F}, n = 1, 2, \ldots,$ disjoint, then $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$.*

Definition 1.1.2. *A probability space $(\Omega, \mathcal{F}, P)$ is complete if $A \in \mathcal{F}$ has $P(A) = 0$ and $B \subset A$, then $B \in \mathcal{F}$ (and, of course, $P(B) = 0$), that is, $\mathcal{F}$ contains all P-null sets.*

Remark 1.1.3. *Any probability space $(\Omega, \mathcal{F}, P)$ can be made complete by the completion of measures due to Carathéodory. We also refer to the proof of the Daniell Theorem, Theorem 2.1 in Chapter 2.*

Definition 1.1.4. *For any collection $\mathcal{G}$ of subsets of Ω, we define a smallest σ-algebra $\sigma(\mathcal{G})$ containing $\mathcal{G}$ by*

$$\sigma(\mathcal{G}) = \cap\{\mathcal{F} : \mathcal{G} \subset \mathcal{F},\ \mathcal{F}\ \sigma\text{-algebra of } \Omega\},$$

which is the σ-algebra generated by $\mathcal{G}$.

Example 1.1.5. *On the set of real numbers $\mathbf{R}$, we take $\mathcal{G} = \{\text{open intervals}\}$ and denote by $\mathcal{B}(\mathbf{R})$ the σ-algebra $\sigma(\mathcal{G})$ generated by $\mathcal{G}$, which is the Borel σ-algebra on $\mathbf{R}$.*

Definition 1.1.6. *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.*

(a) *A map $X : \Omega \to \mathbf{R}$ is a random variable if*

$$X^{-1}(B) := \{\omega : X(\omega) \in B\} \in \mathcal{F}, \qquad \text{for any } B \in \mathcal{B}(\mathbf{R}).$$

(b) *For any random variable X, we define the σ-algebra $\sigma(X)$ generated by X as follows:*

$$\sigma(X) = \sigma(\mathcal{G}) = \mathcal{G}, \qquad \mathcal{G} := \{X^{-1}(B) : B \in \mathcal{B}(\mathbf{R})\} \subset \mathcal{F}.$$

Proposition 1.1.7. *Let X, Y be two random variables. Then Y is $\sigma(X)$ measurable if and only if there exists a Borel measurable function $g : \mathbf{R} \to \mathbf{R}$ such that*

$$Y(\omega) = g(X(\omega)), \qquad \text{for all } \omega \in \Omega.$$

Proof. Since $Y = Y^+ - Y^-$, we will show the "only if" part when $Y \geq 0$.

(1) Suppose that Y is a simple random variable. Then Y is of the form:

$$Y(\omega) = \sum_{i=1}^{n} y_i 1_{F_i}(\omega),$$

where $y_i \geq 0$, $F_i \in \sigma(X)$ and the F_i are pairwise disjoint. By definition, there exists $D_i \in \mathcal{B}(\mathbf{R})$, for each i, such that $F_i = X^{-1}(D_i)$. Clearly, the D_i are pairwise disjoint. Define

$$g(y) = \begin{cases} y_i, & y \in D_i, \\ 0, & y \notin \bigcup_{i=1}^{n} D_i. \end{cases}$$

Then

$$Y(\omega) = \sum_{i=1}^{n} y_i 1_{\{X^{-1}(D_i)\}}(\omega) = \sum_{i=1}^{n} y_i 1_{D_i}(X(\omega)) = g(X(\omega)).$$

(2) In the general case, there exists a sequence of simple random variables Y_n converging to Y. Let g_n be the corresponding sequence of measurable functions such that $Y_n = g_n(X)$. Define

$$g(y) = \liminf_{n\to\infty} g_n(y).$$

Then g is $\mathcal{B}(\mathbf{R})$ measurable and

$$Y(\omega) = \liminf_n Y_n(\omega) = \liminf_n g_n(X(\omega)) = g(X(\omega)), \quad \omega \in \Omega. \quad \blacksquare$$

1.1.2 Expectation, Conditional Expectation

Definition 1.1.8. *Let X be a random variable. The quantity*

$$E[X] = \int_\Omega X(\omega) dP(\omega)$$

is the expectation of X, where $E[X^+]$ or $E[X^-]$ is finite.

Definition 1.1.9. *Let X, Y be two random variables on a complete probability space $(\Omega, \mathcal{F}, P)$.*

(a) *The expression $X = Y$ will indicate that $X = Y$ a.s., that is, $P(X \neq Y) = 0$.*
(b) *For $1 \le p < \infty$, the norm $\|X\|_p$ of X is defined by*

$$\|X\|_p = (E[|X|^p])^{1/p}.$$

(c) *If $p = \infty$, then*

$$\|X\|_\infty = ess\ sup|X| = \inf\{\sup_{\omega\notin N} |X(\omega)| : N \in \mathcal{F}, P(N) = 0\}.$$

(d) *The L^p spaces are defined by*

$$L^p = L^p(\Omega) = \{X : random\ variable, \|X\|_p < \infty\}.$$

Proposition 1.1.10.

(i) *$L^p(\Omega)$ is a Banach space, that is, a complete normed linear space, for $1 \le p \le \infty$.*
(ii) *$L^2(\Omega)$ is a Hilbert space, that is, a complete inner product space, with inner product $(X, Y) = E[X \cdot Y], \quad X, Y \in L^2(\Omega)$.*

For the proof, see A. Friedman [69, chapter 3].

Definition 1.1.11. *Let X_n, $n = 1, 2, \ldots,$ and X be random variables.*

(a) *$X_n \to X$ a.s. if $P(X_n \to X$ as $n \to \infty) = 1$.*

(b) $X_n \to X$ *in probability if* $P(|X_n - X| \geq \varepsilon) \to 0$ *as* $n \to \infty$, *for any* $\varepsilon > 0$.
(c) $X_n \to X$ *in* L^p *if* $\|X_n - X\|_p \to 0$ *as* $n \to \infty$.

Proposition 1.1.12. *Let* $X_n, n = 1, 2, \ldots,$ *and* X *be random variables.*

(i) *If* $X_n \to X$ *a.s., then* $X_n \to X$ *in probability.*
(ii) *If* $X_n \to X$ *in* L^p $(p \geq 1)$, *then* $X_n \to X$ *in probability.*
(iii) $X_n \to X$ *in probability if and only if* $E[\frac{|X_n - X|}{1+|X_n - X|}] \to 0$.
(iv) *Let* $\varepsilon_n \geq 0$ *and* $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. *If* $\sum_{n=1}^{\infty} P(|X_{n+1} - X_n| \geq \varepsilon_n) < \infty$, *then* X_n *converges a.s.*
(v) *If* $X_n \to X$ *in probability, then it contains a subsequence* $\{X_{n_k}\}$ *such that* $X_{n_k} \to X$ *a.s.*

For the proof, see A. Friedman [69, chapter 2].

Definition 1.1.13. *A family* $\{X_n : n \in \mathbf{N}\}$ *of random variables* X_n *on* $(\Omega, \mathcal{F}, P)$ *is uniformly integrable if*

$$\lim_{a \to \infty} \sup_n \int_{\{|X_n| \geq a\}} |X_n| dP = 0.$$

Proposition 1.1.14. *Assume that one of the following assertions is satisfied:*

(i) $E[\sup_n |X_n|] < \infty$,
(ii) $\sup_n E[|X_n|^p] < \infty$, *for some* $p > 1$.

Then $\{X_n\}$ *is uniformly integrable.*

Proof.

(1) We set $Y = \sup_n |X_n|$. Then, by (i),

$$P(Y \geq c) \leq \frac{1}{c} E[Y] \to 0 \quad \text{as } c \to \infty.$$

Thus

$$\sup_n \int_{\{|X_n| \geq c\}} |X_n| dP \leq \int_{\{Y \geq c\}} Y dP \to 0 \quad \text{as } c \to \infty.$$

(2) By Chebyshev's inequality,

$$\sup_n P(|X_n| \geq c) \leq \frac{1}{c^p} \sup_n E[|X_n|^p].$$

Thus, by (ii) and Hölder's inequality,

$$\begin{aligned}\sup_n \int_{\{|X_n| \geq c\}} |X_n| dP &\leq \sup_n (E[|X_n|^p])^{1/p} (E[1_{\{|X_n| \geq c\}}])^{1/q} \\ &\leq \sup_n E[|X_n|^p] (\frac{1}{c})^{p/q} \to 0 \quad \text{as } c \to \infty,\end{aligned}$$

where $1/p + 1/q = 1$. ∎

Proposition 1.1.15. *Let $\{X_n\}$ be a sequence of integrable random variables such that $X_n \to X$ a.s. Then $\{X_n\}$ is uniformly integrable if and only if*

$$\lim_{n\to\infty} E[|X_n - X|] = 0.$$

Proof. Let $Y_n = X_n - X$.

(1) Suppose that $\{X_n\}$ is uniformly integrable. Since

$$\begin{aligned} E[|X_n|] &= E[|X_n|1_{\{|X_n|\geq a\}}] + E[|X_n|1_{\{|X_n|<a\}}] \\ &\leq \sup_n E[|X_n|1_{\{|X_n|\geq a\}}] + aP(|X_n| < a), \quad \text{for any } a > 0, \end{aligned}$$

we have $\sup_n E[|X_n|] < \infty$, taking sufficiently large $a > 0$. By Fatou's lemma

$$E[|X|] \leq \liminf_{n\to\infty} E[|X_n|] < \infty.$$

Also, by Chebyshev's inequality,

$$\begin{aligned} E[|Y_n|] &\leq E[|X_n|1_{\{|Y_n|\geq a\}} + |X|1_{\{|Y_n|\geq a\}} + |Y_n|1_{\{|Y_n|<a\}}] \\ &\leq E[|X_n|1_{\{|X_n|\geq c\}}] + 2cP(|Y_n| \geq a) + E[|X|1_{\{|X|\geq c\}}] \\ &\quad + E[|Y_n|1_{\{|Y_n|<a\}}] \\ &\leq \sup_n E[|X_n|1_{\{|X_n|\geq c\}}] + \frac{2c}{a}\sup_n E[|X_n| + |X|] + E[|X|1_{\{|X|\geq c\}}] \\ &\quad + (1+a)E[\frac{|Y_n|}{1+|Y_n|}]. \end{aligned}$$

Letting $n \to \infty$, $a \to \infty$, and then $c \to \infty$, we get $\limsup_{n\to\infty} E[|Y_n|] = 0$.

(2) Conversely, it is easy to see that

$$P(|Y_n| \geq a) \leq \frac{1}{a}E[|Y_n|] \to 0 \quad \text{as } a \to \infty$$

and

$$\begin{aligned} E[|Y_n|1_{\{|Y_n|\geq a\}}] &= E[(|Y_n| - |Y_n| \wedge m)1_{\{|Y_n|\geq a\}}] + E[(|Y_n| \wedge m)1_{\{|Y_n|\geq a\}}] \\ &\leq E[|Y_n| - |Y_n| \wedge m] + mP(|Y_n| \geq a). \end{aligned}$$

Hence, letting $a \to \infty$ and then $m \to \infty$, we get

$$\limsup_{a\to\infty} E[|Y_n|1_{\{|Y_n|\geq a\}}] = 0, \quad \text{for each } n.$$

Next, for any $k \in \mathbf{N}$,

$$\begin{aligned} \sup_n E[|Y_n|1_{\{|Y_n|\geq a\}}] &\leq \sup_{n\leq k} E[|Y_n|1_{\{|Y_n|\geq a\}}] + \sup_{n>k} E[|Y_n|1_{\{|Y_n|\geq a\}}] \\ &\leq \sum_{n=1}^{k} E[|Y_n|1_{\{|Y_n|\geq a\}}] + \sup_{n>k} E[|Y_n|]. \end{aligned}$$

Letting $a \to \infty$ and then $k \to \infty$, we have $\limsup_{a\to\infty} \sup_n E[|Y_n|1_{\{|Y_n|\geq a\}}] = 0$. By the same calculation as shown in (1), we have

$$\begin{aligned}\sup_n E[|X_n|1_{\{|X_n|\geq c\}}] &\leq \sup_n E[|Y_n|1_{\{|X_n|\geq c\}} + |X|1_{\{|X_n|\geq c\}}]\\ &\leq \sup_n E[|Y_n|1_{\{|Y_n|\geq a\}}] + \frac{2a}{c}\sup_n E[|X_n|]\\ &\quad + E[|X|1_{\{|X|\geq a\}}].\end{aligned}$$

Letting $c \to \infty$ and then $a \to \infty$, we obtain uniform integrability. ■

Definition 1.1.16. *Let $X \in L^1(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, that is, $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra. A random variable $Y \in L^1(\Omega, \mathcal{G}, P)$ is the conditional expectation of X given $\mathcal{G}$ if*

$$\int_A Y dP = \int_A X dP, \quad \text{for all } A \in \mathcal{G}.$$

We write $Y = E[X|\mathcal{G}]$.

Proposition 1.1.17. *Let $X \in L^1(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. Then the conditional expectation $Y \in L^1(\Omega, \mathcal{G}, P)$ of X given $\mathcal{G}$ exists uniquely.*

Proof. Without loss of generality, we may assume $X \geq 0$. Define

$$\mu(A) = \int_A X dP, \quad \text{for } A \in \mathcal{G}.$$

Then μ is a finite measure, absolutely continuous with respect to P. By the Radon–Nikodým Theorem (cf. A. Friedman [69]), there exists, uniquely, $Y \in L^1(\Omega, \mathcal{G}, P)$, $Y \geq 0$, such that $\mu(A) = \int_A Y dP$, for $A \in \mathcal{G}$. ■

Remark 1.1.18. *We recall that $L^2(\Omega, \mathcal{G}, P)$ is a closed subspace of the Hilbert space $L^2(\Omega, \mathcal{F}, P)$. If $X \in L^2(\Omega, \mathcal{F}, P)$, then $E[X|\mathcal{G}]$ coincides with the orthogonal projection $\hat{X}$ of X to $L^2(\Omega, \mathcal{G}, P)$, that is,*

$$E[|X - E[X|\mathcal{G}]|^2] = \min\{E[|X - Y|^2] : Y \in L^2(\Omega, \mathcal{G}, P)\}.$$

Proposition 1.1.19. *Let $X_n, X \in L^1(\Omega, \mathcal{F}, P)$, $n = 1, 2, \ldots,$ and $\mathcal{H}, \mathcal{G}$ be two sub-σ-algebras of $\mathcal{F}$. Then, the following assertions hold:*

(i) $E[E[X|\mathcal{G}]] = E[X]$.
(ii) $E[X|\mathcal{G}] = X$ *a.s. if X is $\mathcal{G}$-measurable.*
(iii) $E[aX_1 + bX_2|\mathcal{G}] = aE[X_1|\mathcal{G}] + bE[X_2|\mathcal{G}]$ *a.s.*, $\quad a, b \in \mathbf{R}$.
(iv) $E[X|\mathcal{G}] \geq 0$ *a.s. if* $X \geq 0$.
(v) $E[X_n|\mathcal{G}] \nearrow E[X|\mathcal{G}]$ *a.s. if* $X_n \nearrow X$.
(vi) $E[\liminf_{n\to\infty} X_n|\mathcal{G}] \leq \liminf_{n\to\infty} E[X_n|\mathcal{G}]$ *a.s. if* $X_n \geq 0$.
(vii) $E[X_n|\mathcal{G}] \to E[X|\mathcal{G}]$ *a.s. if* $X_n \to X$ *a.s. and* $\sup_n |X_n| \in L^1$.
(viii) $f(E[X|\mathcal{G}]) \leq E[f(X)|\mathcal{G}]$ *a.s. if* $f : \mathbf{R} \to \mathbf{R}$ *is convex, and* $f(X) \in L^1(\Omega)$.
(ix) $E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$ *a.s. if* $\mathcal{H} \subset \mathcal{G}$.

(x) $E[ZX|\mathcal{G}] = ZE[X|\mathcal{G}]$ *a.s. if* $X \in L^p(\Omega, \mathcal{F}, P)$, $Z \in L^q(\Omega, \mathcal{G}, P)$, *for* $p = 1$, $q = \infty$ *or* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

(xi) $E[X|\mathcal{G}] = E[X]$ *a.s. if* X *and* 1_A *are independent for any* $A \in \mathcal{G}$.

Proof. The proof is obtained by integrating over arbitrary sets $A \in \mathcal{G}$ and by using several properties of the integrals. In particular, (v) is immediately from the monotone convergence theorem. For (vi), we apply (v) to $Y_n = \inf_{m \geq n} X_m$, and (vii) follows from (vi). ■

1.2 Stochastic Processes

1.2.1 General Notations

Definition 1.2.1. *A quadruple* $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})$ *is a stochastic basis if the following assertions hold:*

(a) $(\Omega, \mathcal{F}, P)$ *is a complete probability space.*

(b) $\{\mathcal{F}_t\}_{t \geq 0}$ *is a filtration, that is, a nondecreasing family of sub-*σ*-algebra of* $\mathcal{F}$:

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \text{ for } 0 \leq s < t < \infty.$$

(c) *The filtration* $\{\mathcal{F}_t\}_{t \geq 0}$ *satisfies the following ("usual") conditions*:

  (i) $\{\mathcal{F}_t\}_{t \geq 0}$ *is right-continuous:* $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$, *for all* $t \geq 0$.

  (ii) $\mathcal{F}_0$ *contains all* P*-null sets in* $\mathcal{F}$.

Definition 1.2.2. *An* N*-dimensional stochastic process* $X = \{X_t\}_{t \geq 0}$ *on a complete probability space* $(\Omega, \mathcal{F}, P)$ *is a collection of* $\mathbf{R}^N$*-valued random variables* $X(t, \omega)$, $\omega \in \Omega$. *For fixed* $\omega \in \Omega$, *the set* $\{X(t, \omega) : t \geq 0\}$ *is a path of* X.

Definition 1.2.3. *Let* $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})$ *be a stochastic basis, and let* $\mathcal{B}(\mathbf{R})$, $\mathcal{B}(\mathbf{R}^N)$, $\mathcal{B}([0, t])$ *be Borel* σ*-algebra on* $\mathbf{R}$, $\mathbf{R}^N$, $[0, t]$.

(a) *A stochastic process* $\{X_t\}_{t \geq 0}$ *is* $\{\mathcal{F}_t\}_{t \geq 0}$*-adapted (with respect to the filtration* $\{\mathcal{F}_t\}_{t \geq 0}$*) if* X_t *is* $\mathcal{F}_t$*-measurable for all* $t \geq 0$.
Such a stochastic process will be denoted by $\{(X_t, \mathcal{F}_t)\}_{t \geq 0}$ *or* $(X_t, \mathcal{F}_t)$, $\{(X(t), \mathcal{F}_t)\}$.

(b) *A stochastic process* $\{X_t\}_{t \geq 0}$ *is measurable if, for all* $A \in \mathcal{B}(\mathbf{R}^N)$,

$$\{(t, \omega) : X_t(\omega) \in A\} \in \mathcal{B}([0, \infty)) \otimes \mathcal{F}.$$

(c) *A measurable adapted stochastic process* $\{X_t\}_{t \geq 0}$ *is progressively measurable if, for each* $t \geq 0$ *and* $A \in \mathcal{B}(\mathbf{R}^N)$,

$$\{(s, \omega) : 0 \leq s \leq t, \ X_s(\omega) \in A\} \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t.$$

(d) *A stochastic process* $\{X_t\}_{t \geq 0}$ *is continuous (right-continuous) if there is* $\Omega_0 \in \mathcal{F}$ *with* $P(\Omega_0) = 1$ *such that* $t \to X_t(\omega)$ *is continuous (right-continuous) for every* $\omega \in \Omega_0$.

(e) *Two stochastic processes X and Y are indistinguishable if there is* $\Omega_0 \in \mathcal{F}$ *with* $P(\Omega_0) = 1$ *such that* $X_t(\omega) = Y_t(\omega)$ *for all* $t \geq 0$ *and* $\omega \in \Omega_0$. *The expression* $X = Y$ *will indicate that X and Y are indistinguishable.*
(f) *Y is a modification of X if* $P(X_t = Y_t) = 1$ *for all* $t \geq 0$.

Proposition 1.2.4. *Let* $\{(X_t, \mathcal{F}_t)\}_{t\geq 0}$ *be a stochastic process on a stochastic basis* $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$.

(i) *If* $X = \{X_t\}$ *is a right-continuous modification of 0, then* $X = 0$.
(ii) *If* $X = \{X_t\}$ *is right-continuous, then X is progressively measurable.*

Proof. Let $\Omega_0 = \{\omega : t \to X_t(\omega) \text{ right-continuous}\}$ and $P(\Omega_0) = 1$.

(1) Since $\{X_t\}$ is a modification of 0, $P(X_r \neq 0) = 0$ for each $r \in \mathbf{Q}_+$, and

$$P(\bigcup_{r\in\mathbf{Q}_+} \{\omega : X_r(\omega) \neq 0\}) = 0.$$

Define $\Omega' = (\cap_{r\in\mathbf{Q}_+} \{\omega : X_r(\omega) = 0\}) \cap \Omega_0$. It is obvious that $P(\Omega') = 1$ and $X_t(\omega) = 0$, for all $t \geq 0$ and $\omega \in \Omega'$.

(2) Taking into account the indistinguishable process of $\{X_t\}$, we may consider that $X(t, \omega) = 0$ if $\omega \notin \Omega_0$. Fix $t \geq 0$, and let $\delta = \{t_0 = 0 < t_1 < t_2 < \cdots < t_n = t\}$ be a partition of $[0, t]$, with $t_k = kt/2^n$, $k = 0, 1, \ldots, 2^n$. Define

$$X^\delta(s, \omega) = \begin{cases} X(0, \omega)1_{\{0\}}(s) + \sum_{k=1}^{2^n} X(t_k, \omega)1_{(t_{k-1}, t_k]}(s) & \text{if } \omega \in \Omega_0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the map $(s, \omega) \to X^\delta(s, \omega)$ is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$-measurable. Letting $n \to \infty$, by right continuity, we see that $X^\delta(s, \omega) \to X(s, \omega)$ for all $s \in [0, t]$ and $\omega \in \Omega$. Thus the map $(s, \omega) \to X(s, \omega)$ is measurable for this σ-algebra, so $\{X_t\}$ is progressively measurable. ■

Remark 1.2.5.

(a) *Let* $\{\mathcal{F}_t\}_{t\geq 0}$ *be a filtration such that* $\mathcal{F}_0$ *contains all P−null sets. If* $X = Y$ *and X is adapted to* $\{\mathcal{F}_t\}_{t\geq 0}$, *then Y is adapted to* $\{\mathcal{F}_t\}_{t\geq 0}$.
(b) *For any filtration* $\{\mathcal{G}_t\}_{t\geq 0}$ *we can obtain the right-continuous filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *defined by* $\mathcal{F}_t = \mathcal{G}_{t+} := \bigcap_{\varepsilon>0} \mathcal{G}_{t+\varepsilon}$.
(c) *The filtration in connection with the stochastic process* $\{X_t\}_{t\geq 0}$ *is the* σ*-algebra*

$\sigma(X_s, s \leq t)$ *generated by* $\{X_s, s \leq t\}$, *where*

$$\sigma(X_s, s \leq t) := \sigma(\mathcal{G}_t), \quad \mathcal{G}_t = \{X_s^{-1}(A) : A \in \mathcal{B}(\mathbf{R}^N), s \leq t\}.$$

(d) *Furthermore, a stochastic basis* $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^X\})$ *is obtained by setting*

$$\mathcal{F}_t^X = \mathcal{H}_{t+}, \quad \mathcal{H}_t = \sigma(X_s, s \leq t) \vee \mathcal{N} := \sigma(\sigma(X_s, s \leq t) \cup \mathcal{N}),$$

where $\mathcal{N}$ *is the collection of P null sets.*

1.2.2 Brownian Motion

Definition 1.2.6.

(a) *The real-valued stochastic process* $\{B_t\}_{t \geq 0}$ *is a (one-dimensional standard) Brownian motion if*
  (i) $\{B_t\}_{t \geq 0}$ *is continuous,* $B_0 = 0$ *a.s.,*
  (ii) $B_t - B_s$ *is independent of* $\mathcal{F}_s^B$,
  (iii) $P(B_t - B_s \in A) = \int_A \frac{1}{\sqrt{2\pi(t-s)}} \exp\{-\frac{x^2}{2(t-s)}\} dx$, *for* $t > s$, $A \in \mathcal{B}(\mathbf{R})$.

(b) *The* N*-dimemsional stochastic process* $B(t) = (B_1(t), B_2(t), \ldots, B_N(t))$ *is a (standard)* N*-dimensional Brownian motion if the* N*-components* $B_i(t)$ *are independent one-dimensional standard Brownian motions.*

Proposition 1.2.7. *A continuous stochastic process* $\{B_t\}_{t \geq 0}$ *with* $B_0 = 0$ *is an* N*-dimensional Brownian motion if and only if*

$$E[e^{i(\xi, B_t - B_s)} | \mathcal{F}_s^B] = e^{-\frac{|\xi|^2}{2}(t-s)}, \quad t > s, \quad \xi \in \mathbf{R}^N,$$

where $i = \sqrt{-1}$, $(\, , \,)$ *denotes the inner product of* $\mathbf{R}^N$, *and* $|\xi| = (\xi, \xi)^{1/2}$.

Proof. For $t > s$, the random variable $Y := B_t - B_s$ is normally distributed with mean 0 and covariance $(t-s)I$ (I : identity), that is to say, the characteristic function of Y is given by

$$E[e^{i(\xi, Y)}] = e^{-\frac{|\xi|^2}{2}(t-s)}.$$

Let Y be independent of $\mathcal{F}_s^B$. Then it is easy to see that

$$E[e^{i(\xi, Y)} | \mathcal{F}_s^B] = E[e^{i(\xi, Y)}].$$

Conversely,

$$E[e^{i(\xi, Y)} e^{i(\eta, Z)}] = E[E[e^{i(\xi, Y)} | \mathcal{F}_s^B] e^{i(\eta, Z)}] = E[e^{i(\xi, Y)}] E[e^{i(\eta, Z)}],$$

for any $\eta \in \mathbf{R}^N$ and $\mathcal{F}_s^B$-measurable random variable Z. Thus we conclude that Y and Z are independent. ■

Remark 1.2.8. *The existence of a Brownian motion can be shown by introducing the probability measure* P, *called a Wiener measure, on the space* $C([0, \infty) : \mathbf{R}^N)$ *of* $\mathbf{R}^N$*-valued contiunuous functions on* $[0, \infty)$. *See I. Karatzas and S. E. Shreve [87] and K. Ito and H. P. Mackean [75] for details. The remarkable properties of the Brownian motion are as follows:*

(a) *The Brownian motion* $\{B_t\}$ *is not differentiable a.s.,*
(b) *The total variation of* $\{B_t\}$ *on* $[0, T]$ *is infinite a.s.*