Contents

Preface *page* xiii

1 Introduction 1

2 Real and complex Wigner matrices 6

2.1 Real Wigner matrices: traces, moments and combinatorics 6

2.1.1 The semicircle distribution, Catalan numbers and Dyck paths 7

2.1.2 Proof #1 of Wigner's Theorem 2.1.1 10

2.1.3 Proof of Lemma 2.1.6: words and graphs 11

2.1.4 Proof of Lemma 2.1.7: sentences and graphs 17

2.1.5 Some useful approximations 21

2.1.6 Maximal eigenvalues and Füredi–Komlós enumeration 23

2.1.7 Central limit theorems for moments 29

2.2 Complex Wigner matrices 35

2.3 Concentration for functionals of random matrices and logarithmic Sobolev inequalities 38

2.3.1 Smoothness properties of linear functions of the empirical measure 38

2.3.2 Concentration inequalities for independent variables satisfying logarithmic Sobolev inequalities 39

2.3.3 Concentration for Wigner-type matrices 42

2.4 Stieltjes transforms and recursions 43

2.4.1 Gaussian Wigner matrices 45

2.4.2 General Wigner matrices 47

2.5 Joint distribution of eigenvalues in the GOE and the GUE 50

2.5.1 Definition and preliminary discussion of the GOE and the GUE 51

2.5.2 Proof of the joint distribution of eigenvalues 54

2.5.3 Selberg's integral formula and proof of (2.5.4) 58

2.5.4 Joint distribution of eigenvalues: alternative formulation 65

2.5.5 Superposition and decimation relations 66

2.6 Large deviations for random matrices 70

2.6.1 Large deviations for the empirical measure 71

2.6.2 Large deviations for the top eigenvalue 81

2.7 Bibliographical notes 85

3 Hermite polynomials, spacings and limit distributions for the Gaussian ensembles 90

3.1 Summary of main results: spacing distributions in the bulk and edge of the spectrum for the Gaussian ensembles 90

3.1.1 Limit results for the GUE 90

3.1.2 Generalizations: limit formulas for the GOE and GSE 93

3.2 Hermite polynomials and the GUE 94

3.2.1 The GUE and determinantal laws 94

3.2.2 Properties of the Hermite polynomials and oscillator wave-functions 99

3.3 The semicircle law revisited 101

3.3.1 Calculation of moments of $\bar{L}_N$ 102

3.3.2 The Harer–Zagier recursion and Ledoux's argument 103

3.4 Quick introduction to Fredholm determinants 107

3.4.1 The setting, fundamental estimates and definition of the Fredholm determinant 107

3.4.2 Definition of the Fredholm adjugant, Fredholm resolvent and a fundamental identity 110

3.5 Gap probabilities at 0 and proof of Theorem 3.1.1 114
3.5.1 The method of Laplace 115
3.5.2 Evaluation of the scaling limit: proof of Lemma 3.5.1 117
3.5.3 A complement: determinantal relations 120
3.6 Analysis of the sine-kernel 121
3.6.1 General differentiation formulas 121
3.6.2 Derivation of the differential equations: proof of Theorem 3.6.1 126
3.6.3 Reduction to Painlevé V 128
3.7 Edge-scaling: proof of Theorem 3.1.4 132
3.7.1 Vague convergence of the largest eigenvalue: proof of Theorem 3.1.4 133
3.7.2 Steepest descent: proof of Lemma 3.7.2 134
3.7.3 Properties of the Airy functions and proof of Lemma 3.7.1 139
3.8 Analysis of the Tracy–Widom distribution and proof of Theorem 3.1.5 142
3.8.1 The first standard moves of the game 144
3.8.2 The wrinkle in the carpet 144
3.8.3 Linkage to Painlevé II 146
3.9 Limiting behavior of the GOE and the GSE 148
3.9.1 Pfaffians and gap probabilities 148
3.9.2 Fredholm representation of gap probabilities 155
3.9.3 Limit calculations 160
3.9.4 Differential equations 170
3.10 Bibliographical notes 181
4 Some generalities 186
4.1 Joint distribution of eigenvalues in the classical matrix ensembles 187
4.1.1 Integration formulas for classical ensembles 187
4.1.2 Manifolds, volume measures and the coarea formula 193

4.1.3 An integration formula of Weyl type 199
4.1.4 Applications of Weyl's formula 206
4.2 Determinantal point processes 214
4.2.1 Point processes: basic definitions 215
4.2.2 Determinantal processes 220
4.2.3 Determinantal projections 222
4.2.4 The CLT for determinantal processes 227
4.2.5 Determinantal processes associated with eigenvalues 228
4.2.6 Translation invariant determinantal processes 232
4.2.7 One-dimensional translation invariant determinantal processes 237
4.2.8 Convergence issues 241
4.2.9 Examples 243
4.3 Stochastic analysis for random matrices 248
4.3.1 Dyson's Brownian motion 249
4.3.2 A dynamical version of Wigner's Theorem 262
4.3.3 Dynamical central limit theorems 273
4.3.4 Large deviation bounds 277
4.4 Concentration of measure and random matrices 281
4.4.1 Concentration inequalities for Hermitian matrices with independent entries 282
4.4.2 Concentration inequalities for matrices with dependent entries 287
4.5 Tridiagonal matrix models and the β ensembles 302
4.5.1 Tridiagonal representation of β ensembles 303
4.5.2 Scaling limits at the edge of the spectrum 306
4.6 Bibliographical notes 318
5 Free probability 322
5.1 Introduction and main results 323
5.2 Noncommutative laws and noncommutative probability spaces 325

5.2.1 Algebraic noncommutative probability spaces and laws 325
5.2.2 C^*-probability spaces and the weak*-topology 329
5.2.3 W^*-probability spaces 339
5.3 Free independence 348
5.3.1 Independence and free independence 348
5.3.2 Free independence and combinatorics 354
5.3.3 Consequence of free independence: free convolution 359
5.3.4 Free central limit theorem 368
5.3.5 Freeness for unbounded variables 369
5.4 Link with random matrices 374
5.5 Convergence of the operator norm of polynomials of independent GUE matrices 394
5.6 Bibliographical notes 410
Appendices 414
A Linear algebra preliminaries 414
A.1 Identities and bounds 414
A.2 Perturbations for normal and Hermitian matrices 415
A.3 Noncommutative matrix L^p-norms 416
A.4 Brief review of resultants and discriminants 417
B Topological preliminaries 418
B.1 Generalities 418
B.2 Topological vector spaces and weak topologies 420
B.3 Banach and Polish spaces 422
B.4 Some elements of analysis 423
C Probability measures on Polish spaces 423
C.1 Generalities 423
C.2 Weak topology 425
D Basic notions of large deviations 427
E The skew field $\mathbb{H}$ of quaternions and matrix theory over $\mathbb{F}$ 430
E.1 Matrix terminology over $\mathbb{F}$ and factorization theorems 431

E.2 The spectral theorem and key corollaries 433
E.3 A specialized result on projectors 434
E.4 Algebra for curvature computations 435
F Manifolds 437
F.1 Manifolds embedded in Euclidean space 438
F.2 Proof of the coarea formula 442
F.3 Metrics, connections, curvature, Hessians, and the Laplace–Beltrami operator 445
G Appendix on operator algebras 450
G.1 Basic definitions 450
G.2 Spectral properties 452
G.3 States and positivity 454
G.4 von Neumann algebras 455
G.5 Noncommutative functional calculus 457
H Stochastic calculus notions 459
References 465
General conventions and notation 481
Index 484