

Cambridge University Press
978-0-521-19440-2 - An Outline of Ergodic Theory
Steven Kalikow and Randall McCutcheon
Frontmatter
[More information](#)

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 122

Editorial Board

B. BOLLOBÁS, W. FULTON, A. KATOK, F. KIRWAN,
P. SARNAK, B. SIMON, B. TOTARO

AN OUTLINE OF ERGODIC THEORY

This informal introduction provides a fresh perspective on isomorphism theory, which is the branch of ergodic theory that explores the conditions under which two measure-preserving systems are essentially equivalent. It contains a primer in basic measure theory, proofs of fundamental ergodic theorems, and material on entropy, martingales, Bernoulli processes, and various varieties of mixing.

Original proofs of classic theorems – including the Shannon–McMillan–Breiman theorem, the Krieger finite generator theorem, and the Ornstein isomorphism theorem – are presented by degrees, together with helpful hints that encourage the reader to develop the proofs on their own. Hundreds of exercises and open problems are also included, making this an ideal text for graduate courses. Professionals needing a quick review, or seeking a different perspective on the subject, will also value this book.

STEVEN KALIKOW is a Visiting Professor in the Department of Mathematical Sciences at the University of Memphis.

RANDALL MCCUTCHEON is Associate Professor in the Department of Mathematical Sciences at the University of Memphis.

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

Editorial Board:

B. Bollobás, W. Fulton, A. Katok, F. Kirwan, P. Sarnak, B. Simon, B. Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit: <http://www.cambridge.org/series/sSeries.asp?code=CSAM>

Already published

- 73 B. Bollobás *Random graphs (2nd Edition)*
- 74 R. M. Dudley *Real analysis and probability (2nd Edition)*
- 75 T. Sheil-Small *Complex polynomials*
- 76 C. Voisin *Hodge theory and complex algebraic geometry, I*
- 77 C. Voisin *Hodge theory and complex algebraic geometry, II*
- 78 V. Paulsen *Completely bounded maps and operator algebras*
- 79 F. Gesztesy & H. Holden *Soliton equations and their algebro-geometric solutions, I*
- 81 S. Mukai *An introduction to invariants and moduli*
- 82 G. Tourlakis *Lectures in logic and set theory, I*
- 83 G. Tourlakis *Lectures in logic and set theory, II*
- 84 R. A. Bailey *Association schemes*
- 85 J. Carlson, S. Müller-Stach & C. Peters *Period mappings and period domains*
- 86 J. J. Duistermaat & J. A. C. Kolk *Multidimensional real analysis, I*
- 87 J. J. Duistermaat & J. A. C. Kolk *Multidimensional real analysis, II*
- 89 M. C. Golumbic & A. N. Trenk *Tolerance graphs*
- 90 L. H. Harper *Global methods for combinatorial isoperimetric problems*
- 91 I. Moerdijk & J. Mrcun *Introduction to foliations and Lie groupoids*
- 92 J. Kollár, K. E. Smith & A. Corti *Rational and nearly rational varieties*
- 93 D. Applebaum *Lévy processes and stochastic calculus (1st Edition)*
- 94 B. Conrad *Modular forms and the Ramanujan conjecture*
- 95 M. Schechter *An introduction to nonlinear analysis*
- 96 R. Carter *Lie algebras of finite and affine type*
- 97 H. L. Montgomery & R. C. Vaughan *Multiplicative number theory, I*
- 98 I. Chavel *Riemannian geometry (2nd Edition)*
- 99 D. Goldfeld *Automorphic forms and L-functions for the group $GL(n, \mathbb{R})$*
- 100 M. B. Marcus & J. Rosen *Markov processes, Gaussian processes, and local times*
- 101 P. Gille & T. Szamuely *Central simple algebras and Galois cohomology*
- 102 J. Bertoin *Random fragmentation and coagulation processes*
- 103 E. Frenkel *Langlands correspondence for loop groups*
- 104 A. Ambrosetti & A. Malchiodi *Nonlinear analysis and semilinear elliptic problems*
- 105 T. Tao & V. H. Vu *Additive combinatorics*
- 106 E. B. Davies *Linear operators and their spectra*
- 107 K. Kodaira *Complex analysis*
- 108 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli *Harmonic analysis on finite groups*
- 109 H. Geiges *An introduction to contact topology*
- 110 J. Faraut *Analysis on Lie groups: An Introduction*
- 111 E. Park *Complex topological K-theory*
- 112 D. W. Stroock *Partial differential equations for probabilists*
- 113 A. Kirillov, Jr *An introduction to Lie groups and Lie algebras*
- 114 F. Gesztesy *et al. Soliton equations and their algebro-geometric solutions, II*
- 115 E. de Faria & W. de Melo *Mathematical tools for one-dimensional dynamics*
- 116 D. Applebaum *Lévy processes and stochastic calculus (2nd Edition)*
- 117 T. Szamuely *Galois groups and fundamental groups*
- 118 G. W. Anderson, A. Guionnet & O. Zeitouni *An introduction to random matrices*
- 119 C. Perez-Garcia & W. H. Schikhof *Locally convex spaces over non-Archimedean valued fields*
- 120 P. K. Friz & N. B. Victoir *Multidimensional stochastic processes as rough paths*
- 121 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli *Representation theory of the symmetric groups*

Cambridge University Press
978-0-521-19440-2 - An Outline of Ergodic Theory
Steven Kalikow and Randall McCutcheon
Frontmatter
[More information](#)

An Outline of Ergodic Theory

STEVEN KALIKOW
RANDALL MCCUTCHEON

University of Memphis



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press
 978-0-521-19440-2 - An Outline of Ergodic Theory
 Steven Kalikow and Randall McCutcheon
 Frontmatter
[More information](#)

CAMBRIDGE UNIVERSITY PRESS
 Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
 São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press
 The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
 Information on this title: www.cambridge.org/9780521194402

© S. Kalikow and R. McCutcheon 2010

This publication is in copyright. Subject to statutory exception
 and to the provisions of relevant collective licensing agreements,
 no reproduction of any part may take place without the written
 permission of Cambridge University Press.

First published 2010

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

Kalikow, Steven, 1950–

An outline of ergodic theory / Steven Kalikow, Randall McCutcheon.

p. cm. – (Cambridge studies in advanced mathematics ; 122)

ISBN 978-0-521-19440-2 (hardback)

1. Ergodic theory. 2. Isomorphisms (Mathematics) I. McCutcheon,
 Randall, 1965– II. Title. III. Series.

QA313.K35 2010

515'.48–dc22

2010000325

ISBN 978-0-521-19440-2 Hardback

Cambridge University Press has no responsibility for the persistence or
 accuracy of URLs for external or third-party internet websites referred to in
 this publication, and does not guarantee that any content on such websites is,
 or will remain, accurate or appropriate.

Contents

<i>Preface</i>	<i>page vii</i>
Introduction	1
1 Measure-theoretic preliminaries	5
1.1 Basic definitions	5
1.2 Carathéodory's theorem, isomorphism, Lebesgue spaces	7
1.3 Properties of Lebesgue spaces, factors	11
1.4 Random variables, integration, (stationary) processes	16
1.5 Conditional expectation	23
2 Measure-preserving systems, stationary processes	26
2.1 Systems and homomorphisms	26
2.2 Constructing measure-preserving transformations	27
2.3 Types of processes; ergodic, independent and (P, T)	30
2.4 Rohlin tower theorem	32
2.5 Countable generator theorem	37
2.6 Birkhoff ergodic theorem and the strong law	39
2.7 Measure from a monkey sequence	43
2.8 Ergodic decomposition	45
2.9 Ergodic theory on L^2	48
2.10 Conditional expectation of a measure	50
2.11 Subsequential limits, extended monkey method	52
3 Martingales and coupling	55
3.1 Martingales	55
3.2 Coupling; the basics	59
3.3 Applications of coupling	62
3.4 The d bar and variation distances	67
3.5 Preparation for the Shannon–Macmillan–Breiman theorem	69
4 Entropy	72
4.1 The 3-shift is not a factor of the 2-shift	72
4.2 The Shannon–McMillan–Breiman theorem	74
4.3 Entropy of a stationary process	78

vi	<i>Contents</i>	
4.4	An abstraction: partitions of 1	79
4.5	Measurable partitions, entropy of measure-preserving systems	82
4.6	Krieger finite generator theorem	85
4.7	The induced transformation and \bar{f}	92
5	Bernoulli transformations	96
5.1	The cast of characters	96
5.2	$\mathbf{FB} \subset \mathbf{IC}$	99
5.3	$\mathbf{IC} \subset \mathbf{EX}$	102
5.4	$\mathbf{FD} \subset \mathbf{IC}$	112
5.5	$\mathbf{EX} \subset \mathbf{VWB}$	115
5.6	$\mathbf{EX} \subset \mathbf{FD}$	121
5.7	$\mathbf{VWB} \subset \mathbf{IC}$	122
6	Ornstein isomorphism theorem	124
6.1	Copying in distribution	124
6.2	Coding	128
6.3	Capturing entropy: preparation	129
6.4	Tweaking a copy to get a better copy	135
6.5	Sinai's theorem	139
6.6	Ornstein isomorphism theorem	143
7	Varieties of mixing	146
7.1	The varieties of mixing	146
7.2	Ergodicity vs. weak mixing	147
7.3	Weak mixing vs. mild mixing	150
7.4	Mild mixing vs. strong mixing	153
7.5	Strong mixing vs. Kolmogorov	155
7.6	Kolmogorov vs. Bernoulli	162
	<i>Appendix</i>	167
	<i>References</i>	170
	<i>Index</i>	173

Preface

This book treats *isomorphism theory* – that branch of ergodic theory dealing with the question of when two measure-preserving systems are, in a certain sense, essentially equivalent. Although these topics have received fair treatment in several books, we think that the time is right for a fresh perspective. Indeed, with ergodic theory becoming more fashionable in its connections with number theory and additive combinatorics, yet also more abstract and structure-laden, it is interesting to observe the extent to which progress in its original concerns, classification of measure-preserving systems up to isomorphism, was achieved via combinatorial/probabilistic reasoning. Our hope is that the ergodic theory revival currently underway will find its way to isomorphism theory, and revitalize it as well.

We have also attempted to write a book that teaches general mathematical thinking in a unique manner. Most graduate level textbooks in pure mathematics provide detailed proofs of theorems followed by exercises. We¹ have tried to write this book in such a way as to make the proofs of the theorems themselves the exercises. Optional details, of which readers may want more or less, may be relegated to footnotes or to sections labeled “Remark” or “Comment”.

Indeed, proofs of major theorems are generally presented twice; once labeled “Idea of proof”, in which the reader is called on to flesh out the argument from a very basic outline, then again with the label “Sketch of proof”, in which more details are given. We consider it important that the reader attempt to work through the “Idea” section before or instead of the “Sketch”. This reading strategy benefits:

- (a) the serious beginner wishing to work out the proofs on their own, with helpful guidance from the book;
- (b) the student wanting a basic overview of ergodic theory, who doesn’t want to be overburdened with notation while trying to understand the ideas;
- (c) the professional mathematician who is familiar with the material, but wants a quick review or different perspective on it.

¹ The first author has circulated a rough precursor to this book as an MS Word document on the Internet for a number of years; the pedagogic philosophy employed here (and all original proofs of major theorems) are due to him. R.M.

Cambridge University Press
978-0-521-19440-2 - An Outline of Ergodic Theory
Steven Kalikow and Randall McCutcheon
Frontmatter
[More information](#)

viii

Preface

This book differs from other books of its type not only in the careful compartmentalization of detail; much in our approach to classical theorems (those of Shannon–McMillan–Breiman and Ornstein, for example) would be considered unorthodox, however presented. Indeed, our aim is not to present “distinctively modern”, “slick” or “book” proofs. Our aim is to present *our* proofs, stumbled upon by active engagement with the subject matter. Our hope is that readers will follow our example, and come to favor their own proofs as well.