Introduction

Note: this introduction is written in an intuitive style, so a scientifically oriented non-mathematician might get something out of it. It is the only part of the book that requires no mathematical expertise.

Question: *What is ergodic theory?*

Let’s start with two examples.

Example 1: Imagine a potentially oddly shaped billiard table having no pockets and a frictionless surface. Part of the table is painted white and part of the table is painted black. A billiard ball is placed in a random spot and shot along a trajectory with a random velocity. You meanwhile are blindfolded and don’t know the shape of the table. However, as the billiard ball careens around, you receive constant updates on when it’s in the black part of the table, and when it’s in the white part of the table. From this information you are to deduce as much as you can about the entire setup: for example, whether or not it is possible that the table is in the shape of a rectangle.

Example 2: (This example is extremely vague by intention.) Imagine you are receiving a sequence of signals from outer space. The signal seems to be in some sense random, but there are recurring patterns whose frequencies are stationary (that is, do not alter over time). We are unable to detect a precise signal but we can encode it by interpreting five successive signals as one signal: unfortunately, this code loses information. Furthermore, we make occasional mistakes. We wish to get as much knowledge as possible about the original process.

Measure preserving transformations. The subject matter encompassing the previous two examples is called ergodic theory. Ergodic theory models situations (like the examples) under the abstraction of *measure-preserving transformations*. To understand that concept, we need to understand what *measures* are and what *transformations* are.

A measure is a concept of size that tells you how big a set is, or, in the language of probability, how probable an event is. (A probability space is a space whose total measure is equal to 1.) It has to act like a notion of size should: the measure of the union of two disjoint sets has to equal the sum of the measures of the sets, for example. A transformation is a way of mapping a space to itself by assigning one point to another. In many modeling applications, the transformation indicates evolution in time: for example, it may map the position and direction of a billiard ball at the current time to the position and direction of the ball one second later.

Ergodic theory is the study of transformations on probability spaces that preserve measure. So, for example, if a set A of points has measure $\frac{1}{3}$, and a transformation T is measure-preserving, then the set of points which are mapped into A by T will also have measure $\frac{1}{3}$. When the measure is interpreted as a probability, the measure-preserving property indicates the stationarity or time-invariance of the expected frequencies of certain events (like the probability that the billiard ball of Example 1 lies in the white part of the table).

Processes. When you apply a transformation over and over, checking after each application whether some event has occurred and recording the result, you get a *process*. The language of processes and the language of transformations are really just two different ways of describing the same thing. You can get a process out of Example 1 if you record at one-second intervals, by writing down either B or W, the location of the moving billiard ball. For example, the output $BBWWB\dots$ represents the ball having been in the black, black, white, white and black areas at times 0, 1, 2, 3 and 4, respectively. You can also get processes out of real world systems, without foreknowledge of any transformation acting. For example, say you record each day at noon whether it's rainy or sunny by writing down R or S. If you did this every day into both the future and the past, you would output a doubly infinite string of Rs and Ss, thus: $\dots SRRS(S)RRRS\dots$ Here the parentheses identify the current day (it is sunny today, will be rainy tomorrow, and was sunny yesterday, etc.).

To transfer this example to the language of transformations, note that the set of all doubly infinite strings of Rs and Ss forms a space, and a natural transformation of this space is the *shift*, which moves time forward one day. (Hence the shift takes the above sequence to $\dots SRRSS(R)RRS\dots$) An appropriate measure can be derived from the probabilities of rain and sun respectively on the various days. This measure will be preserved by the shift precisely when the original process was stationary.

In this book, we will usually study measure-preserving transformations using the language of stationary processes. Here is a summary of the important concepts and theorems we will cover.

(1) *Isomorphism*: Suppose that in Example 1 we were to change which part of the table is painted white and which is painted black. Then you would have a different process. But our new process could end up being equivalent to the original process in the sense that if you know the output of either process infinitely far into both the past and future, it would tell you the

output of the other process. Very roughly, one says that two processes are *isomorphic* when there is a nice way to map outputs of one to outputs of the other so that they determine each other. In general, determining whether there is such a map can be nearly impossible; much of this book is about ways to do it in a few cases.

(2) *Ergodicity*: An *ergodic* process or transformation is one that cannot be expressed as a combination of two simpler processes (or transformations). For example, consider the process that picks a random person and then spits out an enormous sequence of Ls and Rs according to which hand that person uses to twist open all the doorknobs they encounter their whole life. That process is certainly not going to be ergodic because the character of the output will be divided in a predictable way according to whether the person chosen is left-handed or right-handed. Assuming the proportion of left-handed people in the general population to be 0.09, the whole process would then be expressible as 0.09 (left-hand process) + 0.91 (right-hand process).

(3) *Birkhoff ergodic theorem*: When an ergodic transformation is repeatedly applied to form an ergodic process, then with probability 1, the frequency of time an output of that process spends in a given set is the measure of that set, e.g. if the measure of a set is $\frac{1}{3}$, then it will spend (in an asymptotic sense) $\frac{1}{3}$ of the time in that set.

(4) *Rohlin tower theorem*: Fix an arbitrary positive integer, say 678. For any measure-preserving transformation T that does not simply rotate finite sets of points around, you can break almost the whole space into 678 equally sized disjoint sets $A_1, \ldots, A_{678}$ such that if you arrange the sets as the rungs of a ladder, the transformation consists in simply walking up the ladder; that is, $TA_i = A_{i+1}$.

(5) *Shannon–McMillan–Breiman theorem*: Consider an ergodic process that spits out doubly infinite strings of as and bs. If you pick a random doubly infinite string, then with probability 1, when you look at its sequence of finite initial strings (e.g. $a, ab, abb, abba, \ldots$), that sequence will have probabilities that asymptotically approach a fixed rate of exponential decay. Moreover, that rate of decay will not depend on the sequence you choose.

(6) *Entropy*: The exponential rate of decay just mentioned is called the *entropy* of the process. Recall that essentially all of the doubly infinite strings have the same exponential decay rate. Call the ones that do *reasonable names*. Then the number of reasonable names is approximately equal to the reciprocal of this exponentially decaying probability, that is, it is a quantity that increases at a fixed exponential rate. Thus entropy can also be thought of as

being the asymptotic exponential growth rate of the number of reasonable names.

(7) *Kolmogorov entropy invariance*: Any two isomorphic processes must have the same entropy. This provides a quick way to identify two processes as *not* being isomorphic, namely, having different entropies.

(8) *Independent process*: A stationary process on an alphabet in which the next letter to come in the output string is always independent of the ones that came previously is called an *independent process*. For example, repeatedly rolling a die (even a loaded die) gives an independent process.

(9) *Ornstein isomorphism theorem*: Says that two stationary independent processes are isomorphic if and only if they have the same entropy. Indeed, the standard proofs of the theorem say even more, as they give a condition which is natural to check in many cases such that any two processes that are of equal entropy and satisfy the condition must be isomorphic. This has led to the surprising realization that a great many classes of measure-preserving systems that don't seem at all similar to die rolling or coin tossing are in fact isomorphic to independent processes.

The above list spans the core topics of isomorphism theory. In the final chapter of the book, we touch briefly on additional topics in both isomorphism theory and ergodic theory, more broadly construed. In an appendix, we list some of our favorite open problems.

CAMBRIDGE

Cambridge University Press & Assessment
978-0-521-19440-2 — An Outline of Ergodic Theory
Steven Kalikow , Randall McCutcheon
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More Information

1

Measure-theoretic preliminaries

1. Discussion. In this opening chapter, we offer a review of the basic facts we need from measure theory for the rest of the book (it doubles as an introduction to our pedagogic method). For readers seeking a true introduction to the subject, we recommend first perusing, e.g. Folland (1984); experts meanwhile may safely jump to Chapter 2.

2. Comment. When an exercise is given in the middle of a proof, the end of the exercise will be signaled by a dot: •

The conclusion of a proof is signaled by the box sign at the right margin, thus: □

1.1. Basic definitions

In this subchapter, we discuss algebras, σ-algebras, generation of a σ-algebra by a family of subsets, completion with respect to a measure and relevant definitions.

3. Definition. Let Ω be a set. An *algebra* of subsets of Ω is a non-empty collection $\mathcal{A}$ of subsets of Ω that is closed under finite unions and complementation. A *σ-algebra* is a collection $\mathcal{A}$ of subsets of Ω that is closed under countable unions and complementation.

4. Comment. Every algebra of subsets of Ω contains the *trivial algebra* $\{\emptyset, \Omega\}$.

5. Exercise. Let Ω be a set and let $\mathcal{C}$ be a family of subsets of Ω. Show that the intersection of all σ-algebras of subsets of Ω containing $\mathcal{C}$ is itself a σ-algebra.

6. Definition. Let Ω be a set and let $\mathcal{C}$ be a family of subsets of Ω. The *σ-algebra generated by $\mathcal{C}$* is the intersection of all σ-algebras of subsets of Ω containing $\mathcal{C}$.

7. Definition. Let $\mathcal{A}$ be an algebra of subsets of Ω. A *premeasure* on $\mathcal{A}$ is a finitely additive set function p taking $\mathcal{A}$ to the non-negative reals that also never violates countable additivity except for "undefined" cases caused by $\mathcal{A}$ not being a σ-algebra.[2]

If $\mathcal{A}$ is a σ-algebra, then p is called a *measure*.

8. Definition. Let Ω be a set, and let $\mathcal{A}$ be a σ-algebra of subsets of Ω. The pair $(\Omega, \mathcal{A})$ is called a *measurable space*, and the members of $\mathcal{A}$ are called *measurable sets*. Next let μ be a probability measure defined on $\mathcal{A}$; that is, a measure satisfying $\mu(\Omega) = 1$. The triple $(\Omega, \mathcal{A}, \mu)$ is called a *probability space*.

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. An *event* is a measurable set, that is, a member of $\mathcal{A}$. Two events A and B are *independent* if $\mu(A \cap B) = \mu(A)\mu(B)$.

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. A *null set* is a set $A \in \mathcal{A}$ with $\mu(A) = 0$. $\mathcal{A}$ is said to be *complete with respect to* μ, or $(\Omega, \mathcal{A}, \mu)$ is simply said to be *complete*, if all subsets of null sets are measurable (and hence null sets).

9. Exercise. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and suppose that $\mathcal{B}$ is not complete with respect to μ. Let $\mathcal{A} = \{B \cup C : B \in \mathcal{B}$, there exists a null set D with $C \subset D\}$. Extend μ to $\mathcal{A}$ by the rule $\mu(B \cup C) = \mu(B)$ for the relevant cases. Show that this extension is well defined and that $(\Omega, \mathcal{A}, \mu)$ is a complete probability space.

10. Definition. The *completion* of a probability space $(\Omega, \mathcal{B}, \mu)$ is the probability space $(\Omega, \mathcal{A}, \mu)$ constructed in the previous exercise.

11. Definition. If A and B are sets, the *symmetric difference* of A and B is the set of points in A or B but not both. We denote the symmetric difference by $A \triangle B$.

12. Definition. Suppose that $(\Omega, \mathcal{A}, \mu)$ is a complete measure space and suppose that $\mathcal{C} \subset \mathcal{A}$ is a family of measurable sets. We say that $\mathcal{C}$ *generates* $\mathcal{A}$ *mod zero* if for every $A \in \mathcal{A}$ there exists B in the σ-algebra $\mathcal{B}$ generated by $\mathcal{C}$ such that $\mu(A \triangle B) = 0$.

13. Comment. Notice that this does not imply that $\mathcal{B} = \mathcal{A}$.

14. Definition. Let Ω be a set and denote its power set by $\mathcal{P}(\Omega)$. An *outer measure* on Ω is a non-increasing, countably sub-additive set function μ^* from $\mathcal{P}(\Omega)$ to the non-negative reals taking the empty set to zero.[3]

[2] That is to say, $p : \mathcal{A} \to [0, \infty]$ and if $(A_i)_{i=1}^{\infty} \subset \mathcal{A}$ is pairwise disjoint with $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ then $p(A) = \sum_{i=1}^{\infty} p(A_i)$. Notice that $p(\emptyset) = 0$.

[3] That is, $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ with $\mu^*(\emptyset) = 0$, $\mu^*(A) \leq \mu^*(B)$ whenever $A \subset B$, and $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ for any sequence $(A_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$.

1.2. Carathéodory's theorem, isomorphism, Lebesgue spaces

In this subchapter we develop the machinery for constructing probability spaces and determining when they are essentially the same. We use this machinery to construct Lebesgue measure on the unit interval and define Lebesgue spaces to be those spaces isomorphic to an interval space.

15. Convention. If $(x_n)_{n=1}^{\infty}$ is a sequence, we use the notation $(x_n) \subset X$ to relate the fact that $x_n \in X$ for all n.

16. Theorem. *(Carathéodory; see e.g. Folland 1984, Theorem 1.11.) Let Ω be a set, $\mathcal{A}$ an algebra of subsets of Ω and p a premeasure on $\mathcal{A}$ for which $p(\Omega) = 1$. For every $B \subset \Omega$ let*

$$\mu^*(B) = \inf\left\{ \sum_{i=1}^{\infty} p(A_i) : (A_i)_{i=1}^{\infty} \subset \mathcal{A},\ B \subset \bigcup_{i=1}^{\infty} A_i \right\}.$$

Let $\mathcal{B} = \{B \subset \Omega : \mu^*(B) + \mu^*(B^c) = 1\}$. Then μ^* is an outer measure on Ω which agrees with p on $\mathcal{A}$, $\mathcal{B}$ is a σ-algebra containing $\mathcal{A}$, and the restriction μ of μ^* to $\mathcal{B}$ is a measure.

(Proof omitted.)

17. Exercise. Show that the measure space arrived at in an application of Carathéodory's theorem is complete.

We now give a couple of applications of Carathéodory's theorem.

18. Definition. Let Λ be a countable set and let $\Omega = \Lambda^{\mathbf{Z}}$. A *cylinder set* is a subset of Ω you get by specifying values for finitely many (possibly zero) coordinates.[4] The *support* of a cylinder set is the set of coordinates whose values are specified.[5]

19. Example. The set of $(x_i)_{i=-\infty}^{\infty} \in \Omega$ such that $x_0 = a$, $x_{17} = b$ and $x_{-2} = c$ is a cylinder set. The support of this cylinder set is $\{-2, 0, 17\}$.

20. Theorem. *(Tychonoff; see e.g. Folland 1984, Theorem 4.43.) Let X_i be compact topological spaces, $i \in \mathcal{I}$, and let $X = \prod_{i \in \mathcal{I}}$. Then X is compact in the product topology.*

21. Exercise. Prove Tychonoff's theorem. □

[4] To be more precise: for $r \geq 0$ an integer, $f_1, f_2, \ldots, f_r \in \mathbf{Z}$ and $\lambda_1, \ldots, \lambda_r \subset \Lambda$, set $C = C(f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_r) = \{(x_i)_{i=-\infty}^{\infty} \in \Omega : x_{f_j} = \lambda_j, 1 \leq j \leq r\}$. C is a cylinder set.

[5] So the support of the cylinder set defined in the previous footnote is $\{f_1, f_2, \ldots, f_r\}$.

22. Exercise.

(a) Show that the family $\mathcal{A}$ of unions of finite (possibly empty) collections of cylinder sets in $\Omega = \Lambda^{\mathbf{Z}}$ forms an algebra. Then show that if Λ is finite and $(A_i)_{i=1}^{\infty}$ are pairwise disjoint members of $\mathcal{A}$ whose union is a member of $\mathcal{A}$ then only finitely many of the A_i are non-empty. *Hint: Apply Tychonoff's theorem to* $\Lambda^{\mathbf{Z}}$.

(b) (See, e.g. McCutchen 1999, Theorem 3.2.4.) If Λ is finite, any finitely additive set function p from cylinder sets to the non-negative reals[6] taking Ω to 1 is extendable to a premeasure on $\mathcal{A}$ which may thereby be extended to a measure by Carathéodory's theorem.[7]

23. Exercise. Let $\Omega = [0, 1)$ and denote by $\mathcal{A}$ the set of unions of finite (possibly empty), pairwise disjoint families of half-open intervals $[a, b) \subset [0, 1)$. Show that $\mathcal{A}$ is an algebra of sets. Put $p\big([a, b)\big) = b - a$. Show that p has a unique extension to a premeasure on $\mathcal{A}$.

24. Definition. The outer measure μ^* you get by applying Carathéodory's theorem to the premeasure p of the foregoing exercise is called *Lebesgue outer measure,* which we denote by m^*. We denote by $\mathcal{L}$ the σ-algebra $\mathcal{B}$ coming from Carathéodory's theorem; members of $\mathcal{L}$ are called *Lebesgue measurable sets*. The restriction of m^* to $\mathcal{B}$ is called *Lebesgue measure*, which we denote by m.

25. Remark. Although we've only defined Lebesgue measure, Lebesgue measurable sets, etc. here on the unit interval, one can of course extend this to the whole line in the obvious way; readers should convince themselves of this.

26. Definition. For $A \subset \mathbf{R}$, define the *Lebesgue inner measure* of A to be the quantity $m_*(A) = \sup\{m(B) : B \in \mathcal{L}, B \subset S\}$.[8]

27. Exercise. Prove that for any set A and any interval I, $m_*(A) = |I| - m^*(I \setminus A)$.

[6] In other words, if $C_1, \ldots, C_r$ are pairwise disjoint cylinder sets whose union is a cylinder set C, then $p(C) = \sum_{i=1}^{r} p(C_i)$.

[7] It is instructive, and the reader is encouraged, to explore just what such a finitely additive function looks like. For a quick example, suppose that $\Lambda = \{a, d\}$. The cylinder set C_1 that sees the occurrence of "add" at the zero place (that is, $C(0, 1, 2, a, d, d)$) and the cylinder set C_2 that sees "ada" at the zero place are disjoint and their union is the cylinder set C_3 that sees "ad" at the zero place; hence any premeasure p must satisfy $p(C_1) + p(C_2) = p(C_3)$, but these are the only sorts of conditions.

[8] The reader should check that this supremum is a maximum; i.e. it is attained.

28. Exercise. Show that for any Lebesgue measurable set $A \subset [0,1]$, $m(A) = \sup\{m(K) : K \subset A, K \text{ is closed}\}$ and $m(A) = \inf\{m(U) : A \subset U, U \text{ is open}\}$. (We may sometimes say, accordingly, that Lebesgue measure is *inner regular* with respect to closed sets and *outer regular* with respect to open sets.)

29. Definition. Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces. A function $T : \Omega \to \Omega'$ satisfying $T^{-1}A' \in \mathcal{A}$ for every $A' \in \mathcal{A}'$ is said to be $(\mathcal{A}, \mathcal{A}')$*-measurable,* or simply *measurable* when $\mathcal{A}$ and $\mathcal{A}'$ are understood. Let $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \mu')$ be probability spaces. A measurable function $T : \Omega \to \Omega'$ satisfying $\mu(T^{-1}A') = \mu'(A')$ for every $A' \in \mathcal{A}'$ is said to be *measure-preserving.*

30. Theorem. *(Urysohn's lemma; see e.g. Dudley 2002, Lemma 2.6.3.) Let X be a normal topological space, and let A and B be disjoint closed subsets of X. There exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

(Proof omitted.)

31. Exercise. Let $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \mu')$ be measure spaces and suppose that $\mathcal{A}'$ is generated by a family of sets $\mathcal{B}$. Let $T : \Omega \to \Omega'$. Show that:

(a) if $T^{-1}B \in \mathcal{A}$ for every $B \in \mathcal{B}$ then T is measurable;
(b) if $\mu(T^{-1}B) = \mu'(B)$ for every $B \in \mathcal{B}$ then T is measure-preserving.

32. Definition. Let $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \mu')$ be probability spaces and suppose $\pi : \Omega \to \Omega'$ is a measure-preserving transformation. We say that π is a *homomorphism*, or a *factor map*, and that $(\Omega', \mathcal{A}', \mu')$ is a *factor* of $(\Omega, \mathcal{A}, \mu)$.

33. Definition. Let $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \mu')$ be probability spaces and suppose $T : \Omega \to \Omega'$ is a homomorphism. If there exist full measure sets[9] $X \subset \Omega$ and $X' \subset \Omega'$ such that the restriction of T to X is a bijection to X', and $T^{-1} : X' \to X$ is measurable, then we will say that T is an *isomorphism*, and that the spaces $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \mu')$ are *isomorphic*.

34. Comment. When two spaces $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \mu')$ are isomorphic, then, once appropriate null sets are disregarded, they are "essentially the same space". In other words, they are in fact the same, up to relabeling.

35. Exercise. Show that "is isomorphic to" is an equivalence relation and that completeness is an isomorphism invariant.

[9] A set $X \subset \Omega$ is said to be of full measure if $\mu(\Omega \setminus X) = 0$.

36. Convention. Our attitude toward null sets is that they "don't count". Accordingly we will assume that all probability spaces are complete.

Without this convention, $([0,1],\mathcal{L},m)$ would not be isomorphic to $([0,1],\mathcal{B},m)$, where $\mathcal{B}$ is the σ-algebra of Borel sets (that is, the σ-algebra generated by the open sets) for the rather uninteresting (though non-trivial) reason that $\mathcal{L}$ has more null sets than $\mathcal{B}$ does.

37. Definition. An *interval space* consists of an interval $[0,t]$ equipped with Lebesgue measure, where $0 \le t \le 1$, to which are appended countably many points having a combined positive measure $1-t$; *with atoms* if $t < 1$, *without atoms* if $t = 1$.[10]

38. Exercise. Show that an interval space is a complete probability space.

39. Definition. A *Lebesgue space* is a probability space that is isomorphic to some interval space. A Lebesgue space is *non-atomic* if it is isomorphic to $([0,1],\mathcal{L},m)$.

40. Remark. A classic reference in the theory of Lebesgue spaces, axiomatically defined, is Rohlin (1952), though this and most of the literature on axiomatic treatments is fraught with vagueness and ambiguity (if not confusion), due in part to a cavalier attitude toward sets of measure zero. (For an interesting and well-motivated modern axiomatic treatment, see Rudolph (1990).) An arguably more sensible theory of spaces measurably isomorphic to the unit interval is that of regular Borel probability measures on Polish spaces; however, we are choosing to skirt most of the issues entirely by simply defining Lebesgue spaces to be those that are isomorphic to an interval space. (Not all of the issues: see the axiomatic criterion in Theorem 53 below.)

There are tremendous technical advantages to doing analysis on Lebesgue spaces. Following standard ergodic theory practice, we shall deal almost exclusively with non-atomic Lebesgue spaces in this book. Since, in essence, the only non-atomic Lebesgue space is the unit interval, one can be deceived into thinking that this is unduly restrictive. However, the concept is actually quite general. Indeed, just about every probability space you are likely to encounter is Lebesgue or at least has a Lebesgue completion; in particular, spaces derived from completing regular Borel measures on compact metrizable spaces are Lebesgue; non-Lebesgue spaces are pathological examples, generally deriving

[10] So, let $\theta = [0,t] \cup C$, where C is a countable set whose intersection with $[0,t]$ is empty, let $f : C \to [0,1-t]$ be a function satisfying $\sum_{c\in C} f(c) = 1-t$, let $\mathcal{A}$ consist of all $L \cup D$, where L is a Lebesgue measurable subset of $[0,t]$ and D is any subset of C, and for $A = L \cup D \in \mathcal{A}$, put $\mu(A) = m(L) + \sum_{c\in D} f(c)$. $(\theta,\mathcal{A},\mu)$ is an interval space.