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Basic properties of totally positive and strictly totally positive matrices

In this chapter, we introduce some of the notation, basic definitions, and various classic facts and formulæ. Many of the results of this chapter will be used in subsequent chapters. Matrix notation, more especially the notation used for submatrices and minors, is both clumsy and problematic. It definitely takes getting used to. But rest assured that one does eventually get used to it. The medium here is not the message.

In Section 1.2 we consider some simple and less simple operations that preserve total positivity and strict total positivity. Section 1.3 is about nonsingularity and rank. Here we immediately see results that are less than obvious. Finally, in Section 1.4, we present a few basic determinantal inequalities that are valid for totally positive and strictly totally positive matrices.

1.1 Preliminaries

For a positive integer n, and for each $p \in \{1, \ldots, n\}$, we define the simplex

$$I_p^n := \{\mathbf{i} = (i_1, \ldots, i_p) \,:\, 1 \le i_1 < \cdots < i_p \le n\}$$

in $\mathbb{Z}_+^p$. That is, I_p^n denotes the set of strictly increasing sequences of p integers in $\{1, \ldots, n\}$.

We use the following notation to define submatrices and minors of a matrix. If $A = (a_{ij})_{i=1}^{n}{}_{j=1}^{m}$ is an $n \times m$ matrix, then for each $\mathbf{i} \in I_p^n$ and $\mathbf{j} \in I_q^m$ we let

$$A[\mathbf{i}, \mathbf{j}] = A\begin{bmatrix}\mathbf{i}\\ \mathbf{j}\end{bmatrix} = A\begin{bmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_q \end{bmatrix} := (a_{i_k j_\ell})_{k=1}^{p}{}_{\ell=1}^{q}$$

denote the $p \times q$ submatrix of A determined by the rows indexed $i_1, \ldots, i_p$

and columns indexed $j_1, \ldots, j_q$. When $p = q$ then

$$A(\mathbf{i}, \mathbf{j}) = A\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix} = A\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix} := \det\,(a_{i_k j_\ell})_{k,\ell=1}^{p}$$

denotes the associated minor, i.e., the determinant of the submatrix.

This monograph is about totally positive and strictly totally positive matrices. They are defined as follows:

Definition 1.1 An $n \times m$ matrix A is said to be *totally positive* (TP) if all its minors are nonnegative, i.e.,

$$A(\mathbf{i}, \mathbf{j}) = A\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix} \geq 0 \tag{1.1}$$

for all $\mathbf{i} \in I_p^n$, $\mathbf{j} \in I_p^m$, and all $p = 1, \ldots, \min\{n, m\}$. It is said to be *strictly totally positive* (STP) if strict inequality always holds in (1.1).

We use and reuse various classic facts and formulæ. We list some of them here for easy reference.

Cauchy–Binet and pth compound matrices The pth *compound matrix* of the $n \times m$ matrix A is denoted by $A_{[p]}$ and is defined as the $\binom{n}{p} \times \binom{m}{p}$ matrix with entries

$$(A(\mathbf{i}, \mathbf{j}))_{\mathbf{i} \in I_p^n, \mathbf{j} \in I_p^m}$$

where the $\mathbf{i} \in I_p^n$ and $\mathbf{j} \in I_p^m$ are arranged in lexicographic order, i.e., for distinct $\mathbf{i}, \mathbf{k} \in I_p^n$ we set $\mathbf{i} > \mathbf{k}$ if the first nonzero term in the sequence $i_1 - k_1, \ldots, i_p - k_p$ is positive.

Assume $B = CD$, where B is an $n \times m$ matrix, C is an $n \times r$ matrix, and D is an $r \times m$ matrix. The *Cauchy–Binet formula* may be written as follows. For each $\mathbf{i} \in I_p^n$, $\mathbf{j} \in I_p^m$,

$$B(\mathbf{i}, \mathbf{j}) = \sum_{\mathbf{k} \in I_p^r} C(\mathbf{i}, \mathbf{k}) D(\mathbf{k}, \mathbf{j}),$$

i.e.,

$$B\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix} = \sum_{1 \leq k_1 < \cdots < k_p \leq r} C\begin{pmatrix} i_1, \ldots, i_p \\ k_1, \ldots, k_p \end{pmatrix} D\begin{pmatrix} k_1, \ldots, k_p \\ j_1, \ldots, j_p \end{pmatrix},$$

or, alternatively,

$$B_{[p]} = C_{[p]} D_{[p]}.$$

This is, of course, a generalization of the formula for matrix multiplication. For $p = 1$ the above reduces to

$$b_{ij} = \sum_{k=1}^{r} c_{ik} d_{kj} .$$

This Cauchy–Binet formula is valid when $p \leq \min\{n, m, r\}$. For $p > r$ ($p \leq \min\{n, m\}$) the set I_p^r is empty, and in the above sum we set $B(\mathbf{i}, \mathbf{j}) = 0$. This is the "correct convention" as rank $B \leq r$.

For p vectors $\mathbf{u}^1 = (u_{1,1}, \ldots, u_{n,1}), \ldots, \mathbf{u}^p = (u_{1,p}, \ldots, u_{n,p}) \in \mathbb{C}^n$, and each $\mathbf{i} \in I_p^n$ we set

$$\left(\mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^p\right)(\mathbf{i}) = \det \left(u_{i_\ell, j}\right)_{\ell, j=1}^{p} .$$

We consider $\mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^p$ as a vector in $\mathbb{C}^{\binom{n}{p}}$. It is termed the *Grassman product* or *wedge product* or *exterior product* of $\mathbf{u}^1, \ldots, \mathbf{u}^p$. Obviously $\mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^p = \mathbf{0}$ if and only if the $\mathbf{u}^1, \ldots, \mathbf{u}^p$ are linearly dependent. From the Cauchy–Binet formula it easily follows that

$$A_{[p]}(\mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^p) = A\mathbf{u}^1 \wedge \cdots \wedge A\mathbf{u}^p .$$

Sylvester's Determinant Identity Let A be an $n \times m$ matrix, and let

$$(\alpha_1, \ldots, \alpha_p) \in I_p^n \quad \text{and} \quad (\beta_1, \ldots, \beta_p) \in I_p^m .$$

For each $i \in \{1, \ldots, n\} \backslash \{\alpha_1, \ldots, \alpha_p\}$ and $j \in \{1, \ldots, m\} \backslash \{\beta_1, \ldots, \beta_p\}$ we set

$$b_{ij} = A \begin{pmatrix} k_1, \ldots, k_{p+1} \\ \ell_1, \ldots, \ell_{p+1} \end{pmatrix}$$

where $\{k_1, \ldots, k_{p+1}\}$ is the set of integers $\{\alpha_1, \ldots, \alpha_p, i\}$ arranged in natural (increasing) order, and $\{\ell_1, \ldots, \ell_{p+1}\}$ is the set of integers $\{\beta_1, \ldots, \beta_p, j\}$ arranged in natural order. We generally abuse notation by writing

$$b_{ij} = A \begin{pmatrix} \alpha_1, \ldots, \alpha_p, i \\ \beta_1, \ldots, \beta_p, j \end{pmatrix} .$$

But it is **always** to be understood that we have arranged these row and column indices in natural order.

Sylvester's Determinant Identity states that the minors of the $(n - p) \times (m - p)$ matrix $B = (b_{ij})$ satisfy

$$B \begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix} = \left[A \begin{pmatrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_p \end{pmatrix} \right]^{r-1} A \begin{pmatrix} \alpha_1, \ldots, \alpha_p, i_1, \ldots, i_r \\ \beta_1, \ldots, \beta_p, j_1, \ldots, j_r \end{pmatrix} .$$

The submatrix

$$A\begin{bmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_p\end{bmatrix}$$

is called the *pivot block*.

Inverses If $A=(a_{ij})$ is an $n\times n$ nonsingular matrix, then the elements of its inverse $A^{-1}=(c_{ij})$ satisfy

$$c_{ij}=\frac{(-1)^{i+j}A\begin{pmatrix}1,\ldots,\widehat{j},\ldots,n\\ 1,\ldots,\widehat{i},\ldots,n\end{pmatrix}}{A\begin{pmatrix}1,\ldots,n\\ 1,\ldots,n\end{pmatrix}}$$

where we use $\widehat{j}$ to indicate that we have deleted the jth index. Thus, in the numerator above, we have taken the determinant of the submatrix of A obtained by deleting the jth row and ith column. More generally we have

$$A^{-1}\begin{pmatrix}i_1,\ldots,i_p\\ j_1,\ldots,j_p\end{pmatrix}=\frac{(-1)^{\sum_{k=1}^{p}i_k+j_k}A\begin{pmatrix}j_1',\ldots,j_{n-p}'\\ i_1',\ldots,i_{n-p}'\end{pmatrix}}{A\begin{pmatrix}1,\ldots,n\\ 1,\ldots,n\end{pmatrix}}$$

where $i_1<\cdots<i_p$ and $i_1'<\cdots<i_{n-p}'$ are complementary indices in $\{1,\ldots,n\}$, as are the $j_1<\cdots<j_p$ and $j_1'<\cdots<j_{n-p}'$.

Laplace expansion by minors We will make use of the classical Laplace expansion of a determinant given by

$$A\begin{pmatrix}1,\ldots,n\\ 1,\ldots,n\end{pmatrix}=\sum_{1\le j_1<\cdots<j_p\le n}(-1)^{\sum_{r=1}^{p}i_r+j_r}A\begin{pmatrix}i_1,\ldots,i_p\\ j_1,\ldots,j_p\end{pmatrix}A\begin{pmatrix}i_1',\ldots,i_{n-p}'\\ j_1',\ldots,j_{n-p}'\end{pmatrix}.$$

In the above, $i_1<\cdots<i_p$ and $i_1'<\cdots<i_{n-p}'$ are complementary indices in $\{1,\ldots,n\}$, as are the $j_1<\cdots<j_p$ and $j_1'<\cdots<j_{n-p}'$; p is fixed; and the summation is over all ordered p-tuples $j_1<\cdots<j_p$.

Principal minors If A is an $n\times n$ matrix, then its *principal submatrices* are those submatrices of the form

$$A\begin{bmatrix}i_1,\ldots,i_p\\ i_1,\ldots,i_p\end{bmatrix}.$$

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That is to say, principal submatrices are the square submatrices of A, all of whose diagonal elements are diagonal elements of A. The *principal minors* of A are their determinants

$$A\begin{pmatrix} i_1,\ldots,i_p \\ i_1,\ldots,i_p \end{pmatrix}.$$

Determinantal identity The following determinantal identity will prove very useful. We state it here for easy reference. Let A be an $n \times m$ matrix. Let $1 \le i_1 < \cdots < i_r \le n$ and $1 \le j_1 < \cdots < j_{r+1} \le m$. Then for any $k \in \{1,\ldots,r\}$ and $\ell \in \{2,\ldots,r\}$

$$\begin{aligned} &A\begin{pmatrix} i_1,\ldots,i_r \\ j_1,\ldots,\widehat{j_\ell},\ldots,j_{r+1} \end{pmatrix} A\begin{pmatrix} i_1,\ldots,\widehat{i_k},\ldots,i_r \\ j_2,\ldots,j_r \end{pmatrix} \\ &\quad = A\begin{pmatrix} i_1,\ldots,i_r \\ j_2,\ldots,j_{r+1} \end{pmatrix} A\begin{pmatrix} i_1,\ldots,\widehat{i_k},\ldots,i_r \\ j_1,\ldots,\widehat{j_\ell},\ldots,j_r \end{pmatrix} \\ &\quad\quad + A\begin{pmatrix} i_1,\ldots,i_r \\ j_1,\ldots,j_r \end{pmatrix} A\begin{pmatrix} i_1,\ldots,\widehat{i_k},\ldots,i_r \\ j_2,\ldots,\widehat{j_\ell},\ldots,j_{r+1} \end{pmatrix}. \end{aligned} \tag{1.2}$$

Proof Let B be the $(r+1) \times (r+1)$ matrix given by

$$B = \begin{bmatrix} a_{i_1j_1} & \cdots & a_{i_1j_r} & a_{i_1j_{r+1}} \\ \vdots & \ddots & \vdots & \vdots \\ a_{i_rj_1} & \cdots & a_{i_rj_r} & a_{i_rj_{r+1}} \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Apply Sylvester's Determinant Identity, where the pivot block of size $(r-1) \times (r-1)$ is given by all but the kth and last row, and all but the first and ℓth column of B. □

1.2 Building (strictly) totally positive matrices

If A is a totally positive or strictly totally positive matrix, then there are other (strictly) totally positive matrices associated with A and derived from A. There are also various operations that preserve the class of totally positive and strictly totally positive matrices. We review some of these here.

The first series of propositions as presented here are easily verified. Their proofs are left to the reader.

Proposition 1.2 *Assume A is a (strictly) totally positive matrix. Then A^T (the transpose of A), as well as every submatrix of A and A^T is (strictly) totally positive.*

Proposition 1.3 *Assume A is an $n \times m$ (strictly) totally positive matrix. Let B denote the matrix obtained from A by reversing the order of both its rows and columns, i.e., if $A = (a_{ij})_{i=1}^{n}{}_{j=1}^{m}$, then $B = (b_{ij})_{i=1}^{n}{}_{j=1}^{m}$, where $b_{ij} = a_{n+1-i,m+1-j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. The matrix B is (strictly) totally positive.*

This next proposition immediately follows from an application of the Cauchy–Binet formula.

Proposition 1.4 *If A is an $n \times m$ totally positive matrix and B an $m \times r$ totally positive matrix, then AB is an $n \times r$ totally positive matrix. If $m \geq \min\{n, r\}$, A is an $n \times m$ strictly totally positive matrix and B an $m \times r$ totally positive matrix of rank r, then AB is an $n \times r$ strictly totally positive matrix. Similarly, if $m \geq \min\{n, r\}$, A is an $n \times m$ totally positive matrix of rank n and B an $m \times r$ strictly totally positive matrix, then AB is an $n \times r$ strictly totally positive matrix.*

Note that if $m < \min\{n, r\}$, then rank $AB \leq m$ and so AB cannot possibly be strictly totally positive.

Proposition 1.5 *The following operations preserve the class of (strictly) totally positive matrices.*

(i) *Multiplying a row (column) by a positive scalar.*
(ii) *Adding a positive multiple of a row (column) to the preceding or the succeeding row (column).*
(iii) *Adding a positive value to the $(1, 1)$ entry of the matrix (and to the (n, m) entry for an $n \times m$ matrix).*

From the formulæ for minors of the inverse we also have the following.

Proposition 1.6 *Assume A is a square strictly totally positive matrix. Then $DA^{-1}D$ is a strictly totally positive matrix, where D is the diagonal matrix with diagonal entries alternately 1 and -1. If A is a nonsingular totally positive matrix, then $DA^{-1}D$ is a nonsingular totally positive matrix.*

In addition, from Sylvester's Determinant Identity we have the following.

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Proposition 1.7 *Assume A is an $n \times m$ (strictly) totally positive matrix. Fix $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_k \le m$, and let*

$$b_{ij} = A\begin{pmatrix} i_1, \ldots, i_k, i \\ j_1, \ldots, j_k, j \end{pmatrix}$$

for $i \in \{1, \ldots, n\} \backslash \{i_1, \ldots, i_k\}$ and $j \in \{1, \ldots, m\} \backslash \{j_1, \ldots, j_k\}$. (Recall that it is to be understood that we have arranged these row and column indices in natural order.) Then $B = (b_{ij})$ is an $(n-k) \times (m-k)$ (strictly) totally positive matrix.

A result that looks similar is the following:

Theorem 1.8 *Assume A is an $n \times m$ totally positive matrix. Given k, set*

$$b_{ij} = a_{ij} A\begin{pmatrix} 1, \ldots, k \\ 1, \ldots, k \end{pmatrix} - A\begin{pmatrix} 1, \ldots, k, i \\ 1, \ldots, k, j \end{pmatrix}$$

for $i = k+1, \ldots, n$, $j = k+1, \ldots, m$. Then $B = (b_{ij})$ is an $(n-k) \times (m-k)$ totally positive matrix. Furthermore, rank $B \le k$, and if A is a strictly totally positive matrix then all $r \times r$ minors of B are strictly positive for $r = 1, \ldots, \min\{k, n-k, m-k\}$.

We defer the proof of this theorem to the end of this chapter. There are additional operations that preserve strict total positivity and total positivity, but they are not as obvious or as immediate as some of the previous operations. We list two of them here, but defer their proof to Section 2.2.

Proposition 1.9 *Assume A is an $n \times m$ strictly totally positive matrix. For given $1 \le r < n$ set*

$$c_{ij} = \begin{cases} a_{ij}, & i = 1, \ldots, r, \;\; j = 2, \ldots, m \\ A\begin{pmatrix} i-1, i \\ 1, j \end{pmatrix}, & i = r+1, \ldots, n, \;\; j = 2, \ldots, m. \end{cases}$$

Then $C = (c_{ij})$ is an $n \times (m-1)$ strictly totally positive matrix. If A is a totally positive matrix, then C is a totally positive matrix.

Proposition 1.10 *Assume A is an $n \times m$ strictly totally positive matrix. Fix $p < \min\{m, n\}$ and set*

$$c_{ij} = A\begin{pmatrix} i-p, \ldots, i-1, i \\ 1, \ldots, p, j \end{pmatrix}$$

for $i = p+1, \ldots, n$, $j = p+1, \ldots, m$. Then $C = (c_{ij})$ is an $(n-p) \times (m-p)$ strictly totally positive matrix. If A is a totally positive matrix, then C is a totally positive matrix.

When $p = 1$, the above matrix C is a submatrix of the matrix C (with $r = 1$) in Proposition 1.9.

A totally different operation that preserves total positivity and strict total positivity is the following form of iteration. Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix. We define the matrix $B = (b_{ij})_{i,j=1}^n$ as follows:

$$b_{1j} = a_{1j}, \qquad j = 1, \ldots, n,$$

and for $i \geq 2$

$$b_{ij} = \sum_{k=1}^n b_{i-1,k} a_{kj}, \qquad j = 1, \ldots, n.$$

Theorem 1.11 *Let A and B be as defined above. If A is a (strictly) totally positive matrix, then B is a (strictly) totally positive matrix.*

Proof We prove the theorem assuming A is strictly totally positive. The same proof holds, verbatim, for A totally positive.

Our proof will use induction arguments. From the definition we see that $b_{ij} > 0$ for all i, j. Assume that we have proven that all $p \times p$ minors of B are strictly positive for $p = 1, \ldots, r-1$. We prove that the same holds for all $r \times r$ minors.

Given $1 \leq i_1 < \cdots < i_r \leq n$, $1 \leq j_1 < \cdots < j_r \leq n$, consider

$$B \begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix}.$$

We first assume that $i_1 = 1$. Expanding the above minor by its first row we obtain

$$B \begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix} = \sum_{s=1}^r (-1)^{s-1} a_{1j_s} B \begin{pmatrix} i_2, \ldots, i_r \\ j_1, \ldots, \widehat{j_s}, \ldots, j_r \end{pmatrix}.$$

As $1 < i_2 < \cdots < i_r \leq n$, it follows from the Cauchy–Binet formula that

$$B \begin{pmatrix} i_2, \ldots, i_r \\ j_1, \ldots, \widehat{j_s}, \ldots, j_r \end{pmatrix} = \sum_{1 \leq k_1 < \cdots < k_{r-1} \leq n} B \begin{pmatrix} i_2 - 1, \ldots, i_r - 1 \\ k_1, \ldots, k_{r-1} \end{pmatrix} A \begin{pmatrix} k_1, \ldots, k_{r-1} \\ j_1, \ldots, \widehat{j_s}, \ldots, j_r \end{pmatrix}.$$

Thus

$$B\begin{pmatrix} i_1,\ldots,i_r \\ j_1,\ldots,j_r \end{pmatrix} =$$
$$\sum_{s=1}^{r}(-1)^{s-1}a_{1j_s}\sum_{1\le k_1<\cdots<k_{r-1}\le n} B\begin{pmatrix} i_2-1,\ldots,i_r-1 \\ k_1,\ldots,k_{r-1} \end{pmatrix}$$
$$A\begin{pmatrix} k_1,\ldots,k_{r-1} \\ j_1,\ldots,\widehat{j_s},\ldots,j_r \end{pmatrix}$$
$$= \sum_{1\le k_1<\cdots<k_{r-1}\le n} B\begin{pmatrix} i_2-1,\ldots,i_r-1 \\ k_1,\ldots,k_{r-1} \end{pmatrix}\sum_{s=1}^{r}(-1)^{s-1}a_{1j_s}$$
$$A\begin{pmatrix} k_1,\ldots,k_{r-1} \\ j_1,\ldots,\widehat{j_s},\ldots,j_r \end{pmatrix}$$
$$= \sum_{2\le k_1<\cdots<k_{r-1}\le n} B\begin{pmatrix} i_2-1,\ldots,i_r-1 \\ k_1,\ldots,k_{r-1} \end{pmatrix} A\begin{pmatrix} 1,k_1,\ldots,k_{r-1} \\ j_1,\ldots,j_r \end{pmatrix}.$$

As each of the factors in the last sum is strictly positive (we use here the induction hypothesis) we have that

$$B\begin{pmatrix} i_1,\ldots,i_r \\ j_1,\ldots,j_r \end{pmatrix} > 0.$$

We complete the proof, for this fixed r, by applying an induction argument based on the value i_1. We have proved the result for $i_1 = 1$. Now assume that $i_1 > 1$. From the Cauchy–Binet formula,

$$B\begin{pmatrix} i_1,\ldots,i_r \\ j_1,\ldots,j_r \end{pmatrix} = \sum_{1\le k_1<\cdots<k_r\le n} B\begin{pmatrix} i_1-1,\ldots,i_r-1 \\ k_1,\ldots,k_r \end{pmatrix} A\begin{pmatrix} k_1,\ldots,k_r \\ j_1,\ldots,j_r \end{pmatrix}.$$

By our assumption on A and induction hypothesis on B each factor in the sum is strictly positive. This proves the theorem. □

This next result is interesting as we only vary columns and we only consider $n \times n$ minors of A (and thus do not really need the full total positivity of A to obtain our result).

Let A be an $n \times m$ matrix where $n < m$, and assume that

$$A\begin{pmatrix} 1,\ldots,n \\ 1,\ldots,n \end{pmatrix} \neq 0.$$

We define the $n \times (m-n)$ matrix $B = (b_{ij})$ as follows:

$$b_{ij} = A\begin{pmatrix} 1,\ldots,n \\ 1,\ldots,\widehat{i},\ldots,n,n+j \end{pmatrix}, \qquad i = 1,\ldots,n,\ j = 1,\ldots,m-n.$$

Then,

Theorem 1.12 *For A and B as defined above,*

$$B\begin{pmatrix} i_1,\ldots,i_r \\ j_1,\ldots,j_r \end{pmatrix} = (-1)^{\frac{r(r-1)}{2}}\left[A\begin{pmatrix} 1,\ldots,n \\ 1,\ldots,n \end{pmatrix}\right]^{r-1} A\begin{pmatrix} 1,\ldots,n \\ i'_1,\ldots,i'_{n-r},n+j_1,\ldots,n+j_r \end{pmatrix}$$

where $i'_1,\ldots,i'_{n-r}$ is complementary to $i_1,\ldots,i_r$ in $\{1,\ldots,n\}$. Set $C = (c_{ij})_{i=1}^{n}{}_{j=1}^{m-n}$ where

$$c_{ij} = b_{n-i+1,j}, \qquad i=1,\ldots,n,\ j=1,\ldots,m-n.$$

If A is a (strictly) totally positive matrix, then C is a (strictly) totally positive matrix.

Proof Let D be the $2n\times m$ matrix whose first n rows are the unit vectors $\mathbf{e}^i$, $i=1,\ldots,n$, and whose last n rows are A. We apply Sylvester's Determinant Identity with pivot block

$$D\begin{bmatrix} n+1,\ldots,2n \\ 1,\ldots,n \end{bmatrix}.$$

That is, set

$$e_{ij} = D\begin{pmatrix} i,n+1,\ldots,2n \\ 1,\ldots,n,n+j \end{pmatrix}, \qquad i=1,\ldots,n,\ j=1,\ldots,m-n.$$

Note that

$$e_{ij} = (-1)^{i+1}A\begin{pmatrix} 1,\ldots,n \\ 1,\ldots,\widehat{i},\ldots,n,n+j \end{pmatrix} = (-1)^{i+1}b_{ij}.$$

Therefore, from Sylvester's Determinant Identity,

$$\begin{aligned} B\begin{pmatrix} i_1,\ldots,i_r \\ j_1,\ldots,j_r \end{pmatrix} &= (-1)^{r+\sum_{k=1}^{r} i_k} E\begin{pmatrix} i_1,\ldots,i_r \\ j_1,\ldots,j_r \end{pmatrix} \\ &= (-1)^{r+\sum_{k=1}^{r} i_k}\left[D\begin{pmatrix} n+1,\ldots,2n \\ 1,\ldots,n \end{pmatrix}\right]^{r-1} D\begin{pmatrix} i_1,\ldots,i_r,n+1,\ldots,2n \\ 1,\ldots,n,n+j_1,\ldots,n+j_r \end{pmatrix}. \end{aligned}$$

Now

$$D\begin{pmatrix} n+1,\ldots,2n \\ 1,\ldots,n \end{pmatrix} = A\begin{pmatrix} 1,\ldots,n \\ 1,\ldots,n \end{pmatrix}$$