Introduction

Two developments in the late 1960s and early 1970s set the stage for supergravity. First the standard model took shape and was decisively confirmed by experiments. The key theoretical concept underlying this progress was *gauge symmetry*, the idea that symmetry transformations act independently at each point of spacetime. In the standard model these are internal symmetries, whose parameters are Lorentz scalars $\theta^A(x)$ that are *arbitrary functions* of the spacetime point x. These parameters are coordinates of the compact Lie group SU(3) ⊗ SU(2) ⊗ U(1). Scalar, spinor, and vector fields of the theory are each classified in representations of this group, and the Lagrangian is invariant under group transformations. The special dynamics associated with the non-abelian gauge principle allows different realizations of the symmetry in the particle spectrum and interactions that would be observed in experiments. For example, part of the gauge symmetry may be 'spontaneously broken'. In the standard model this produces the 'unification' of weak and electromagnetic interactions. The observed strength and range of these forces are very different, yet the gauge symmetry gives them a common origin.

The other development was global (also called rigid) supersymmetry [1, 2, 3]. It is the unique framework that allows fields and particles of different spin to be unified in representations of an algebraic system called a superalgebra. The symmetry parameters are spinors ϵ_α that are *constant*, independent of x. The simplest $\mathcal{N} = 1$ superalgebra contains a spinor supercharge Q_α and the energy–momentum operator P_a. The anti-commutator of two supercharges is a translation in spacetime. The $\mathcal{N} = 1$ supersymmetry algebra has representations containing massless particles of spins $(s, s - 1/2)$ for $s = 1/2, 1, \ldots$ and somewhat larger representations containing particles with a common non-vanishing mass. Thus supersymmetry always unites bosons, integer spin, with fermions, half-integer spin. The focus of early work was interacting field theories of the $(1/2, 0)$ and $(1, 1/2)$ multiplets. It was found that the ultraviolet divergences of supersymmetric theories are less severe than in the standard model due to the cancelation between bosons and fermions in loop diagrams.

Unbroken supersymmetry requires a spectrum of particles in equal-mass boson–fermion pairs. This is decidedly not what is observed in experiments. So if supersymmetry is realized in Nature, it must appear as a broken symmetry. Through the years much theoretical effort has been devoted to the construction of extensions of the standard model with broken supersymmetry. It is hypothesized that the as yet unseen superpartners of the known particles will be produced at the Large Hadron Collider (LHC) accelerator, thus confirming a supersymmetric version of the standard model. The advantages of supersymmetric models include the following:

- Milder ultraviolet divergences permit an improved and more predictive treatment of electroweak symmetry breaking.
- When extrapolated using the renormalization group, the three distinct gauge couplings of the standard model approach a common value at high energy. The unification of couplings is a major success.
- Supersymmetry provides natural candidates for the particles of cosmological cold dark matter.

The role of gauge symmetry in the standard model suggested that a gauged form of supersymmetry would be interesting and perhaps more powerful than the global form. Such a theory would contain gauge fields for both spacetime translations P_a and SUSY transformations generated by Q_α. Thus gauged supersymmetry was expected to be an extension of general relativity in which the graviton acquires a fermionic partner called the gravitino. The name supergravity is certainly appropriate and was used even before the theory was actually found. It was reasonable to think that the gauge fields of the theory would be the vierbein, $e_\mu^a(x)$, needed to describe gravity coupled to fermions, and a vector–spinor field, $\psi_{\mu\alpha}(x)$, for the gravitino. The graviton and gravitino belong to the $(2, 3/2)$ representation of the algebra. A Lagrangian field theory of supergravity was formulated in the spring of 1976 in [4]. The approach taken was to modify the known free field Lagrangian for $\psi_{\mu\alpha}$ to agree with gravitational gauge symmetry and then find, by a systematic procedure, the additional terms necessary for invariance under supersymmetry transformations with arbitrary $\epsilon_\alpha(x)$. Soon an alternative approach appeared [5] in which the most complicated calculation required in [4] is avoided.

Research in supergravity became a very intense activity in the years following its discovery. One early direction was the construction of Lagrangian field theories in which the spin-(2, 3/2) gravity multiplet is coupled to the $(1/2, 0)$ and $(1, 1/2)$ multiplets of global supersymmetry. This is the framework of matter-coupled supergravity. It shares the positive features of global symmetry listed above. In addition supergravity provides new scenarios for the breaking of supersymmetry. In particular, the structure of the supergravity Lagrangians allows SUSY breaking with vanishing vacuum energy and thus vanishing cosmological constant. This feature is not available without the coupling of matter fields to supergravity. Matter-coupled supergravity theories typically contain scalar fields, which can be useful in constructing phenomenological models of inflationary cosmology.

A spin-3/2 particle is the key prediction of supergravity. SUSY breaking gives it a mass whose magnitude depends on the breaking mechanism. Unfortunately it appears difficult to detect it at the LHC because it is coupled to matter with the feeble strength of quantum gravity. However, gravitinos can be copiously produced in the ultra-high-temperature environment at or near the big bang. Gravitino production leads to important constraints on early universe cosmology.

A second direction of research involves the construction of theories with several supercharges $Q_{i\alpha}$, $i = 1, 2, \ldots, \mathcal{N}$. Such extended supergravity theories can be constructed up to the limit $\mathcal{N} = 8$ in spacetime dimension $D = 4$. Beyond that the superalgebra representations necessarily contain particles of spin $s \geq 5/2$, for which no consistent interactions exist. Many of the ultraviolet divergences expected in a field theory containing gravity are known to cancel in the maximal $\mathcal{N} = 8$ theory, and some theorists speculate that it is ultraviolet finite to all orders in perturbation theory.

Supergravity theories in spacetime dimensions $D > 4$ have been constructed up to the bound $D = 11$ (which is again due to the higher spin consistency problem). Two 10-dimensional supergravity theories, known as the Type IIA and Type IIB theories, are related to the superstring theories that carry the same names. Roughly speaking, supergravity appears as the low-energy limit of superstring theory. This means that the dynamics of the lowest-energy modes of the superstring are described by supergravity. But these statements do not do justice to the intimate and rich relation of these two theoretical frameworks.

The very important anti-de Sitter/conformal field theory (AdS/CFT) correspondence provides one example of this relation. It was based on the remarkable conjecture that Type IIB string theory on the product manifold $\mathrm{AdS}_5 \otimes S^5$ is equivalent to the maximal $\mathcal{N} = 4$ global supersymmetric gauge theory. However, concrete tests and predictions of AdS/CFT usually involve working in the limit in which classical supergravity is a valid approximation to string theory.

The scope of supergravity is broad. There is a supergravity-inspired approach to positive energy and stability in gravitational theories. Many classical solutions of supergravity have the special Bogomol'nyi–Prasad–Sommerfield (BPS) property and therefore satisfy tractable first order field equations. The scalar sectors of supergravity theories involve non-linear σ-models on complex manifolds with new geometries of interest in both physics and mathematics.

To summarize: supergravity is based on the gauge principle of local supersymmetry and is thus connected to fundamental ideas in theoretical physics. Supergravity effects may turn out to be observable at the LHC. Further there is important input from cosmology. This real side of the subject is far from confirmation, but it must be taken seriously. In addition there are several more theoretical applications such as BPS solutions and AdS/CFT. Active research continues on most branches of supergravity although 35 years have passed since it was first formulated.

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PART I

RELATIVISTIC FIELD THEORY IN MINKOWSKI SPACETIME

CAMBRIDGE

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1 Scalar field theory and its symmetries

The major purpose of the early chapters of this book is to review the basic notions of relativistic field theory that underlie our treatment of supergravity. In this chapter we discuss the implementation of internal and spacetime symmetries using the model of a system of free scalar fields as an example. The general Noether formalism for symmetries is also discussed. Our book largely involves classical field theory. However, we adopt conventions for symmetries that are compatible with implementation at the quantum level.

Our treatment is not designed to teach the material to readers who are encountering it for the first time. Rather we try to gather the ideas (and the formulas!) that are useful background for later chapters. Supersymmetry and supergravity are based on symmetries such as the spacetime symmetry of the Poincaré group and much more!

As in much of this book, we assume general spacetime dimension D, with special emphasis on the case $D = 4$.

1.1 The scalar field system

We consider a system of n real scalar fields $\phi^i(x)$, $i = 1, \ldots, n$, that propagate in a flat spacetime whose metric tensor

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-, +, \ldots, +) \tag{1.1}$$

describes one time and $D - 1$ space dimensions. This is Minkowski spacetime, in which we use Cartesian coordinates x^μ, $\mu = 0, 1, \ldots, D - 1$, with time coordinate $x^0 = t$ (with velocity of light $c = 1$).

Practicing physicists and mathematicians are largely concerned with fields that satisfy nonlinear equations. However, linear wave equations, which describe free relativistic particles, have much to teach about the basic ideas. We therefore assume that our fields satisfy the Klein–Gordon equation

$$\Box\phi^i(x) = m^2\phi^i(x)\,, \tag{1.2}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the Lorentz invariant d'Alembertian wave operator.

The equation has plane wave solutions $e^{\pm i(\vec{p}\cdot\vec{x}-Et)}$, which provide the wave functions for particles of spatial momentum $\vec{p}$, with spatial components p^i, and energy $E = p^0 = \sqrt{\vec{p}^2 + m^2}$. The general solution of the equation is the sum of a positive frequency part, which can be expressed as the $(D-1)$-dimensional Fourier transform in the plane waves $e^{i(\vec{p}\cdot\vec{x}-Et)}$, plus a negative frequency part, which is the Fourier transform in the $e^{-i(\vec{p}\cdot\vec{x}-Et)}$,

$$\begin{aligned}\phi^i(x) &= \phi^i_+(x) \;+\; \phi^i_-(x)\,,\\ \phi^i_+(x) &= \int \frac{d^{D-1}\vec{p}}{(2\pi)^{(D-1)}2E} e^{i(\vec{p}\cdot\vec{x}-Et)} a^i(\vec{p})\,,\\ \phi^i_-(x) &= \int \frac{d^{D-1}\vec{p}}{(2\pi)^{(D-1)}2E} e^{-i(\vec{p}\cdot\vec{x}-Et)} a^{i*}(\vec{p}).\end{aligned} \tag{1.3}$$

In the classical theory the quantities $a^i(\vec{p}),\ a^{i*}(\vec{p})$ are simply complex valued functions of the spatial momentum $\vec{p}$. After quantization one arrives at the true quantum field theory[1] in which $\mathbf{a}^i(\vec{p}),\ \mathbf{a}^{i*}(\vec{p})$ are annihilation and creation operators[2] for the particles described by the field operator $\phi^i(\vec{x})$.

The Klein–Gordon equation (1.2) is the variational derivative $\delta S/\delta\phi^i(x)$ of the action

$$S = \int d^D x\, \mathcal{L}(x) = -\tfrac{1}{2}\int d^D x \left[\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^i + m^2\phi^i\phi^i\right]. \tag{1.4}$$

The repeated index i is summed. The action is a *functional* of the fields $\phi^i(x)$. It is a real number that depends on the configuration of the fields throughout spacetime.

1.2 Symmetries of the system

Consider a set of fields such as the $\phi^i(x)$ that satisfy equations of motion such as (1.2). A general symmetry of the system is a mapping of the configuration space, $\phi^i(x) \to \phi'^i(x)$, with the property that if the original field configuration $\phi^i(x)$ is a solution of the equations of motion, then the transformed configuration $\phi'^i(x)$ is also a solution. For scalar fields and for most other systems of interest in this book, we can restrict attention to symmetry transformations that leave the action invariant. Thus we require that the mapping has the property[3,4]

$$S[\phi^i] = S[\phi'^i]. \tag{1.5}$$

Here is an example.

[1] When desirable for clarity we use bold face to indicate the operator in the quantum theory that corresponds to a given classical quantity.

[2] In the conventions above, creation and annihilation operators are normalized in the quantum theory by $[\mathbf{a}(\vec{p}),\ \mathbf{a}^*(\vec{p}')] = (2\pi)^{D-1}2E\,\delta(\vec{p}-\vec{p}')$.

[3] Such mappings must also respect the boundary conditions. This requirement can be non-trivial, e.g. Neumann and Dirichlet boundary conditions for the bosonic string lead to different spacetime symmetry groups. We will mostly assume that field configurations vanish at large spacetime distances.

[4] One important exception is the electromagnetic duality symmetry, which is discussed in Sec. 4.2.

Exercise 1.1 *Verify that the map $\phi^i(x) \to \phi'^i(x) = \phi^i(x+a)$ satisfies (1.5) if a^μ is a constant vector. This symmetry is called a global spacetime translation.*

We consider both spacetime symmetries, which involve a motion in Minkowski spacetime such as the global translation of the exercise, and internal symmetries, which do not. Internal symmetries are simpler to describe, so we start with them.

1.2.1 SO(n) internal symmetry

Let $R^i{}_j$ be a matrix of the orthogonal group SO(n). This means that it is an $n \times n$ matrix that satisfies

$$R^i{}_k \delta_{ij} R^j{}_\ell = \delta_{k\ell}\,, \qquad \det R = 1. \tag{1.6}$$

It is quite obvious that the linear map,

$$\phi^i(x) \to \phi'^i(x) = R^i{}_j \phi^j(x)\,, \tag{1.7}$$

satisfies (1.5) and is an internal symmetry of the action (1.4). This symmetry is called a continuous symmetry because a matrix of SO(n) depends continuously on $\frac{1}{2}n(n-1)$ independent group parameters. We will discuss one useful choice of parameters shortly. We also call the symmetry a global symmetry because the parameters are constants. In Ch. 4 we will consider local or gauged internal symmetries in which the group parameters are arbitrary functions of x^μ.

It is worth stating the intuitive picture of this symmetry. One may consider the field ϕ^i as an n-dimensional vector, that is an element of $\mathbb{R}^n$. The transformation $\phi^i \to R^i{}_j\phi^j$ is a rotation in this internal space. Such a rotation preserves the usual norm $\phi^i \delta_{ij} \phi^j$.

We now introduce the Lie algebra of the group SO(n). To first order in the small parameter ϵ, we write the infinitesimal transformation

$$R^i{}_j = \delta^i_j - \epsilon r^i{}_j. \tag{1.8}$$

This satisfies (1.6) if $r^i{}_j = -r^j{}_i$. Any antisymmetric matrix $r^i{}_j$ is called a generator of SO(n). The *Lie algebra* is the linear space spanned by the $\frac{1}{2}n(n-1)$ independent generators, with the commutator product

$$[r, r'] = r\,r' - r'r. \tag{1.9}$$

Note that matrices are multiplied[5] as $r^i{}_k r'^k{}_j$.

A useful basis for the Lie algebra is to choose generators that act in each of the $\frac{1}{2}n(n-1)$ independent 2-planes of $\mathbb{R}^n$. For the 2-plane in the directions $\hat{i}\,\hat{j}$ this generator is given by

$$r_{[\hat{i}\hat{j}]}{}^i{}_j \equiv \delta^i_{\hat{i}} \delta_{\hat{j}j} - \delta^i_{\hat{j}} \delta_{\hat{i}j} = -r_{[\hat{j}\hat{i}]}{}^i{}_j. \tag{1.10}$$

[5] Some mathematical readers may initially be perturbed by the indices used to express many equations in this book. We will follow the standard conventions used in physics. Unless ambiguity arises we use the Einstein summation convention for repeated indices, usually one downstairs and one upstairs. The summation convention incorporates the standard rules of matrix multiplication.

Note the distinction between the coordinate plane labels in brackets with hatted indices and the row and column indices. The commutators of the generators defined in (1.10) are

$$[r_{[\hat{i}\hat{j}]}, r_{[\hat{k}\hat{l}]}] = \delta_{\hat{j}\hat{k}} r_{[\hat{i}\hat{l}]} - \delta_{\hat{i}\hat{k}} r_{[\hat{j}\hat{l}]} - \delta_{\hat{j}\hat{l}} r_{[\hat{i}\hat{k}]} + \delta_{\hat{i}\hat{l}} r_{[\hat{j}\hat{k}]}. \tag{1.11}$$

The row and column indices are suppressed in this equation, and this will be our practice when it causes no ambiguity. The equation implicitly specifies the structure constants of the Lie algebra in the basis of (1.10).

In this basis, a finite transformation of SO(n) is determined by a set of $\frac{1}{2}n(n-1)$ real parameters $\theta^{\hat{i}\hat{j}}$ which specify the angles of rotation in the independent 2-planes. A general element of (the connected component) of the group can be written as an exponential

$$R = \mathrm{e}^{-\frac{1}{2}\theta^{\hat{i}\hat{j}} r_{[\hat{i}\hat{j}]}}. \tag{1.12}$$

1.2.2 General internal symmetry

It will be useful to establish the notation for the general situation of linearly realized internal symmetry under an arbitrary connected Lie group G, usually a compact group, of dimension dim G. We will be interested in an n-dimensional representation of G in which the generators of its Lie algebra are a set of $n \times n$ matrices $(t_A)^i{}_j$, $A = 1, 2, \ldots, \dim G$. Their commutation relations are[6]

$$[t_A, t_B] = f_{AB}{}^C t_C\,, \tag{1.13}$$

and the $f_{AB}{}^C$ are structure constants of the Lie algebra. The representative of a general element of the Lie algebra is a matrix Θ that is a superposition of the generators with real parameters θ^A, i.e.

$$\Theta = \theta^A t_A. \tag{1.14}$$

An element of the group is represented by the matrix exponential

$$U(\Theta) = \mathrm{e}^{-\Theta} = \mathrm{e}^{-\theta^A t_A}. \tag{1.15}$$

We consider a set of scalar fields $\phi^i(x)$ which transforms in the representation just described. The fields may be real or complex. If complex, the complex conjugate of every element is also included in the set. A group transformation acts by matrix multiplication on the fields:

$$\phi^i(x) \to \phi'^i(x) \equiv U(\Theta)^i{}_j \phi^j(x). \tag{1.16}$$

[6] Although it is common in the physics literature to insert the imaginary i in the commutation rule, we do not do this in order to eliminate 'i's in most of the formulas of the book. This means that compact generators t_A in this book are anti-hermitian matrices.