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1

Vectors

1.1 Definitions (basic)

There are many ways to define a vector. For starters, here's the most basic:

A vector is the mathematical representation of a physical entity that may be characterized by size (or "magnitude") and direction.

In keeping with this definition, speed (how fast an object is going) is not represented by a vector, but velocity (how fast and *in which direction* an object is going) does qualify as a vector quantity. Another example of a vector quantity is force, which describes how strongly and in what direction something is being pushed or pulled. But temperature, which has magnitude but no direction, is not a vector quantity.

The word "vector" comes from the Latin *vehere* meaning "to carry;" it was first used by eighteenth-century astronomers investigating the mechanism by which a planet is "carried" around the Sun.[1] In text, the vector nature of an object is often indicated by placing a small arrow over the variable representing the object (such as $\vec{F}$), or by using a bold font (such as $\boldsymbol{F}$), or by underlining (such as $\underline{F}$ or $\underset{\sim}{F}$). When you begin hand-writing equations involving vectors, it's very important that you get into the habit of denoting vectors using one of these techniques (or another one of your choosing). The important thing is not *how* you denote vectors, it's that you don't simply write them the same way you write non-vector quantities.

A vector is most commonly depicted graphically as a directed line segment or an arrow, as shown in Figure 1.1(a). And as you'll see later in this section, a vector may also be represented by an ordered set of N numbers,

[1] The *Oxford English Dictionary*. 2nd ed. 1989.

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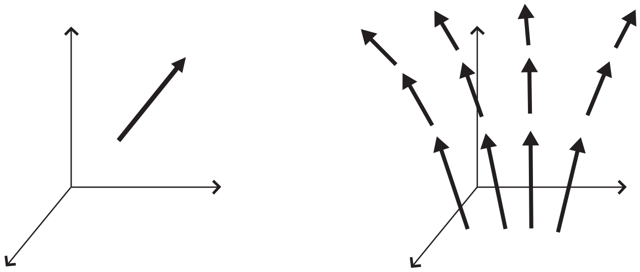

(a) (b)

Figure 1.1 Graphical depiction of a vector (a) and a vector field (b).

where N is the number of dimensions in the space in which the vector resides.

Of course, the true value of a vector comes from knowing what it represents. The vector in Figure 1.1(a), for example, may represent the velocity of the wind at some location, the acceleration of a rocket, the force on a football, or any of the thousands of vector quantities that you encounter in the world every day. Whatever else you may learn about vectors, you can be sure that every one of them has two things: size and direction. The magnitude of a vector is usually indicated by the length of the arrow, and it tells you the amount of the quantity represented by the vector. The scale is up to you (or whoever's drawing the vector), but once the scale has been established, all other vectors should be drawn to the same scale. Once you know that scale, you can determine the magnitude of any vector just by finding its length. The direction of the vector is usually given by indicating the angle between the arrow and one or more specified directions (usually the "coordinate axes"), and it tells you which way the vector is pointing.

So if vectors are characterized by their magnitude and direction, does that mean that two equally long vectors pointing in the same direction could in fact be considered to be the same vector? In other words, if you were to move the vector shown in Figure 1.1(a) to a different location without varying its length or its pointing direction, would it still be the same vector? In some applications, the answer is "yes," and those vectors are called free vectors. You can move a free vector anywhere you'd like as long as you don't change its length or direction, and it remains the same vector. But in many physics and engineering problems, you'll be dealing with vectors that apply *at a given location*; such vectors are called "bound" or "anchored" vectors, and you're not allowed to

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relocate bound vectors as you can free vectors.[2] You may see the term "sliding" vectors used for vectors that are free to move along their length but are not free to change length or direction; such vectors are useful for problems involving torque and angular motion.

You can understand the usefulness of bound vectors if you think about an application such as representing the velocity of the wind at various points in the atmosphere. To do that, you could choose to draw a bound vector at each point of interest, and each of those vectors would show the speed and direction of the wind at that location (most people draw the vector with its tail – the end without the arrow – at the point to which the vector is bound). A collection of such vectors is called a vector field; an example is shown in Figure 1.1(b).

If you think about the ways in which you might represent a bound vector, you may realize that the vector can be defined simply by specifying the start and end points of the arrow. So in a three-dimensional Cartesian coordinate system, you only need to know the values of x, y, and z for each end of the vector, as shown in Figure 1.2(a) (you can read about vector representation in non-Cartesian coordinate systems later in this chapter).

Now consider the special case in which the vector is anchored to the origin of the coordinate system (that is, the end without the arrowhead is at the point of intersection of the coordinate axes, as shown in Figure 1.2(b).[3] Such vectors may be completely specified simply by listing the three numbers that represent the x-, y-, and z-coordinates of the vector's end point. Hence a vector anchored to the origin and stretching five units along the x-axis may be represented as

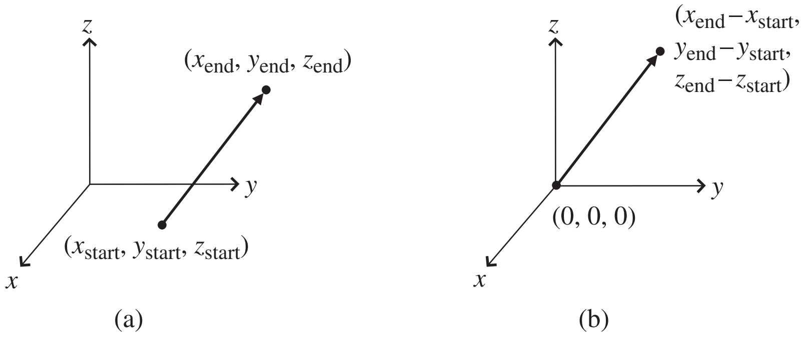

Figure 1.2 A vector in 3-D Cartesian coordinates.

[2] Mathematicians don't have much use for bound vectors, since the mathematical definition of a vector deals with how it transforms rather than where it's located.

[3] The vector shown in Figure 1.2 (a) can be shifted to this location by subtracting x_{start}, y_{start}, and z_{start} from the values at each end.

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(5,0,0). In this representation, the values that represent the vector are called the "components" of the vector, and the number of components it takes to define a vector is equal to the number of dimensions in the space in which the vector exists. So in a two-dimensional space a vector may be represented by a pair of numbers, and in four-dimensional spacetime vectors may appear as lists of four numbers. This explains why a horizontal list of numbers is called a "row vector" and a vertical list of numbers is called a "column vector" in computer science. The number of values in such vectors tells you how many dimensions there are in the space in which the vector resides.

To understand how vectors are different from other entities, it may help to consider the nature of some things that are clearly *not* vectors. Think about the temperature in the room in which you're sitting – at each point in the room, the temperature has a value, which you can represent by a single number. That value may well be different from the value at other locations, but at any given point the temperature can be represented by a single number, the magnitude. Such magnitude-only quantities have been called "scalars" ever since W.R. Hamilton referred to them as "all values contained on the one scale of progression of numbers from negative to positive infinity."[4] Thus

A scalar is the mathematical representation of a physical entity that may be characterized by magnitude only.

Other examples of scalar quantities include mass, charge, energy, and speed (defined as the magnitude of the velocity vector). It is worth noting that the *change* in temperature over a region of space does have both magnitude and direction and may therefore be represented by a vector, so it's possible to produce vectors from groups of scalars. You can read about just such a vector (called the "gradient" of a scalar field) in Chapter 2.

Since scalars can be represented by magnitude only (single numbers) and vectors by magnitude and direction (three numbers in three-dimensional space), you might suspect that there are other entities involving magnitude and directions that are more complex than vectors (that is, requiring more numbers than the number of spatial dimensions). Indeed there are, and such entities are called "tensors."[5] You can read about tensors in the last three chapters of this book, but for now this simple definition will suffice:

[4] W.R. Hamilton, *Phil. Mag.* XXIX, 26.
[5] As you can learn in the later portions of this book, scalars and vectors also belong to the class of objects called tensors but have lower rank, so in this section the word "tensors" refers to higher-rank tensors.

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A tensor is the mathematical representation of a physical entity that may be characterized by magnitude and multiple directions.

An example of a tensor is the inertia that relates the angular velocity of a rotating object to its angular momentum. Since the angular velocity vector has a direction and the angular momentum vector has a (potentially different) direction, the inertia tensor involves multiple directions.

And just as a scalar may be represented by a single number and a vector may be represented by a sequence of three numbers in 3-dimensional space, a tensor may be represented by an array of 3^R numbers in 3-dimensional space. In this expression, "R" represents the rank of the tensor. So in 3-dimensional space, a second-rank tensor is represented by $3^2 = 9$ numbers. In N-dimensional space, scalars still require only one number, vectors require N numbers, and tensors require N^R numbers.

Recognizing scalars, vectors, and tensors is easy once you realize that a scalar can be represented by a single number, a vector by an ordered set of numbers, and a tensor by an array of numbers. So in three-dimensional space, they look like this:

Scalar	Vector	Tensor (Rank 2)
(x)	(x_1, x_2, x_3) or $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$

Note that scalars require no subscripts, vectors require a single subscript, and tensors require two or more subscripts – the tensor shown here is a tensor of rank 2, but you may also encounter higher-rank tensors, as discussed in Chapter 5. A tensor of rank 3 may be represented by a three-dimensional array of values.

With these basic definitions in hand, you're ready to begin considering the ways in which vectors can be put to use. Among the most useful of all vectors are the Cartesian unit vectors, which you can read about in the next section.

1.2 Cartesian unit vectors

If you hope to use vectors to solve problems, it's essential that you learn how to handle situations involving more than one vector. The first step in that process is to understand the meaning of special vectors called "unit vectors" that often

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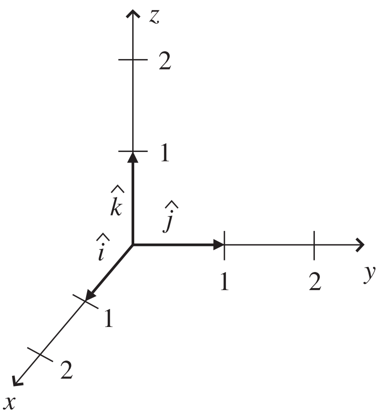

Figure 1.3 Unit vectors in 3-D Cartesian coordinates.

serve as markers for various directions of interest (unit vectors may also be called "versors").

The first unit vectors you're likely to encounter are the unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ (also called $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$) that point in the direction of the x-, y-, and z-axes of the three-dimensional Cartesian coordinate system, as shown in Figure 1.3. These vectors are called unit vectors because their length (or magnitude) is always exactly equal to unity, which is another name for "one." One what? One of whatever units you're using for that axis.

You should note that the Cartesian unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ can be drawn at any location, not just at the origin of the coordinate system. This is illustrated in Figure 1.4. As long as you draw a vector of unit length pointing in the same direction as the direction of the (increasing) x-axis, you've drawn the $\hat{\imath}$ unit vector. So the Cartesian unit vectors show you the directions of the x, y, and z axes, *not* the location of the origin.

As you'll see in Chapter 2, unit vectors can be extremely helpful when doing certain operations such as specifying the portion of a given vector pointing in a certain direction. That's because unit vectors don't have their own magnitude to throw into the mix (actually, they do have their own magnitude, but it is always one).

So when you see an expression such as "$5\hat{\imath}$," you should think "5 units along the positive x-direction." Likewise, $-3\hat{\jmath}$ refers to 3 units along the negative y-direction, and $\hat{k}$ indicates one unit along the positive z-direction.

Of course, there are other coordinate systems in addition to the three perpendicular axes of the Cartesian system, and unit vectors exist in those coordinate

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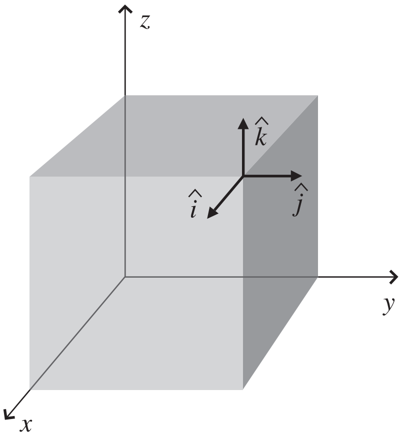

Figure 1.4 Cartesian unit vectors at an arbitrary point.

systems as well; you can see some examples in Section 1.5. One advantage of the Cartesian unit vectors is that they point in the same direction no matter where you go; the x-, y-, and z-axes run in straight lines all the way out to infinity, and the Cartesian unit vectors are parallel to the directions of those lines everywhere.

To put unit vectors such as $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ to work, you need to understand the concept of vector components. The next section shows you how to represent vectors using unit vectors and vector components.

1.3 Vector components

The unit vectors described in the previous section are especially useful when they become part of the "components" of a vector. And what are the components of a vector? Simply stated, they are the pieces that can be used to make up the vector.

To understand vector components, think about the vector $\vec{A}$ shown in Figure 1.5. This is a bound vector, anchored at the origin and extending to the point ($x = 0$, $y = 3$, $z = 3$) in a three-dimensional Cartesian coordinate system. So if you consider the coordinate axes as representing the corner of a room, this vector is embedded in the back wall (the yz plane).

Imagine you're trying to get from the beginning of vector $\vec{A}$ to the end – the direct route would be simply to move in the direction of the vector. But if you were constrained to move only in the directions of the axes, you could get from

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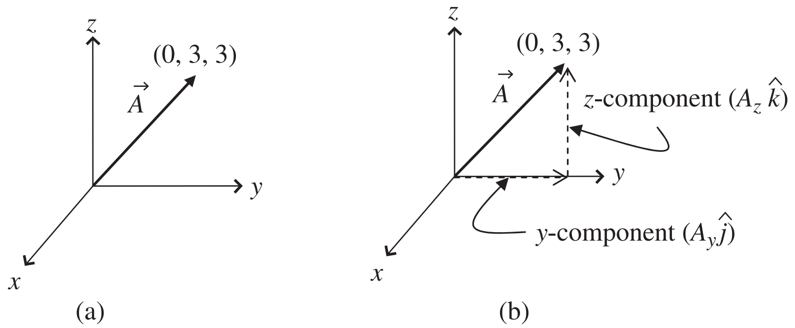

Figure 1.5 Vector $\vec{A}$ and its components.

the origin to your destination by taking three (unit) steps along the y-axis, then turning 90° to your left, and then taking three more (unit) steps in the direction of the z-axis.

What does this little journey have to do with the components of a vector? Simply this: the lengths of the components of vector $\vec{A}$ are the distances you traveled in the directions of the axes. Specifically, in this case the magnitude of the y-component of vector $\vec{A}$ (written as A_y) is just the distance you traveled in the direction of the y-axis (3 units), and the magnitude of the z-component of vector $\vec{A}$ (written as A_z) is the distance you traveled in the direction of the z-axis (also 3 units). Since you didn't move at all in the direction of the x-axis, the magnitude of the x-component of vector $\vec{A}$ (written as A_x) is zero.

A very handy and compact way of writing a vector as a combination of vector components is this:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}, \tag{1.1}$$

where the magnitudes of the vector components (A_x, A_y, and A_z) tell you how many unit steps to take in each direction ($\hat{\imath}, \hat{\jmath}$, and $\hat{k}$) to get from the beginning to the end of vector $\vec{A}$.[6]

When you read about vectors and vector components, you're likely to run across statements such as "The components of a vector are the projections of the vector onto the coordinate axes." As you can see in Chapter 4, exactly how those projections are made can have a significant influence on the nature of the components you get. But in Cartesian coordinate systems (and other

[6] Some authors refer to the magnitudes A_x, A_y, and A_z as the "components of $\vec{A}$," while others consider the components to be $A_x\hat{\imath}$, $A_y\hat{\jmath}$, and $A_z\hat{k}$. Just remember that A_x, A_y, and A_z are scalars, but $A_x\hat{\imath}$, $A_y\hat{\jmath}$, and $A_z\hat{k}$ are vectors.

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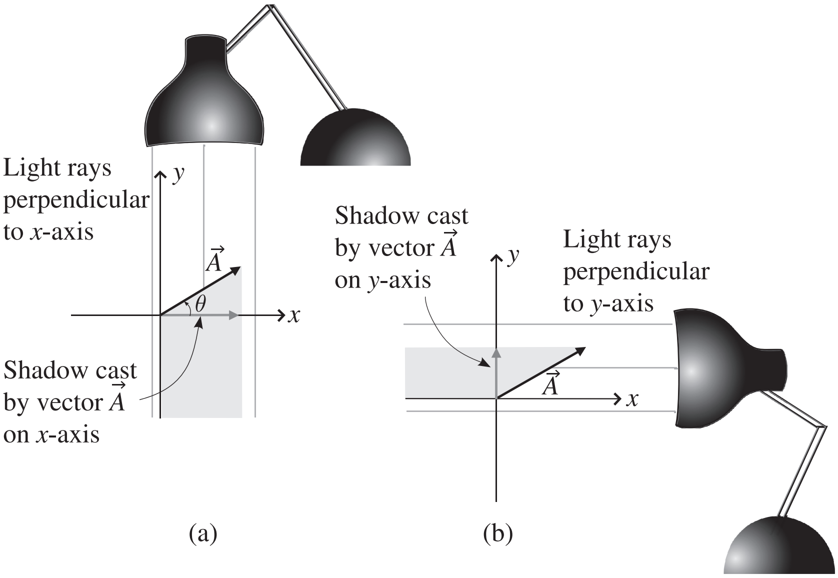

Figure 1.6 Vector components as projections onto x- and y-axes.

"orthogonal" systems in which the axes are perpendicular to one another), the concept of projection onto the coordinate axes is unambiguous and may be very helpful in picturing the components of a vector.

To understand how this works, take a look at vector $\vec{A}$ and the light sources and shadows in Figure 1.6. As you can see in Figure 1.6(a), the direction of the light that produces the shadow on the x-axis is parallel to the y-axis (actually antiparallel since it's moving in the negative y-direction), which in this case is the same as saying that the direction of the light is perpendicular to the x-axis. Likewise, in Figure 1.6(b), the direction of the light that produces the shadow on the y-axis is antiparallel to the x-axis, which is of course perpendicular to the y-axis. This may seem like a trivial point, but when you encounter non-orthogonal coordinate systems, you'll find that the direction parallel to one axis is not necessarily perpendicular to another axis, which gives rise to an entirely different type of vector component. This simple fact has profound implications for the behavior of vectors and tensors for observers in different reference frames, as you'll see in Chapters 4, 5, and 6.

No such issues arise in the two-dimensional Cartesian coordinate system shown in Figure 1.6, and in this case the magnitudes of the components of vector $\vec{A}$ are easy to determine. If the angle between vector $\vec{A}$ and the positive x-axis is θ, as shown in Figure 1.6a, it's clear that the length of $\vec{A}$ can be seen

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as the hypotenuse of a right triangle. The sides of that triangle along the x- and y-axes are the components A_x and A_y. Hence by simple trigonometry you can write:

$$\begin{aligned} A_x &= |\vec{A}|\cos(\theta), \\ A_y &= |\vec{A}|\sin(\theta), \end{aligned} \tag{1.2}$$

where the vertical bars on each side of $\vec{A}$ signify the magnitude (length) of vector $\vec{A}$. Notice that so long as you measure the angle θ *from the positive x-axis* in the direction toward the positive y-axis (that is, counter-clockwise in this case), these equations will give the correct sign for the x- and y-components no matter which quadrant the vector occupies.

For example, if vector $\vec{A}$ is a vector with a length of 7 meters pointing in a direction 210° counter-clockwise from the $+x$-axis, the x- and y-components are given by Eq. 1.2 as

$$\begin{aligned} A_x &= |\vec{A}|\cos(\theta) = 7\text{m } \cos 210^\circ = -6.1\text{ m}, \\ A_y &= |\vec{A}|\sin(\theta) = 7\text{m } \sin 210^\circ = -3.5\text{ m}. \end{aligned} \tag{1.3}$$

As expected for a vector pointing down and to the left from the origin, both components are negative.

It's equally straightforward to find the length and direction of a vector if you're given the vector's Cartesian components. Since the vector forms the hypotenuse of a right triangle with sides A_x and A_y, the Pythagorean theorem tells you that the length of $\vec{A}$ must be

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}, \tag{1.4}$$

and from trigonometry

$$\theta = \arctan\left(\frac{A_y}{A_x}\right), \tag{1.5}$$

where θ is measured counter-clockwise from the positive x-axis in a right-handed coordinate system. If you try this with the components of vector $\vec{A}$ from Eq. 1.3 and end up with a direction of 30° rather than 210°, remember that unless you have a four-quadrant arctan function on your calculator, you must add 180° to the angle whenever the denominator of the expression (A_x in this case) is negative.

Once you have a working understanding of unit vectors and vector components, you're ready to do basic vector operations. The entirety of Chapter 2 is devoted to such operations, but two of them are needed for the remainder of this chapter. For that reason, you can read about vector addition and multiplication by a scalar in the next section.

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