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Overview: Main Themes. Key Issues. Reader's Guide

In this overview we wish to place the body of work described in this book in perspective, connecting it with subject matters in other works sharing similar goals but pursued in different ways or operating at different levels of inquiry. There are many such subjects, but two groups stand out as quite obvious or timely. The obvious subject is quantum gravity (QG) (see, e.g., [1, 2, 3, 4, 5] and [6, 7, 8] for a sampling of the different schools of thought), pursued in the last seventy years, while the timely one is gravitational quantum physics (GQP) (e.g., [9]) a recently minted term presenting a new emphasis on two old disciplines. Quantum gravity refers to theories of the microscopic structures of spacetime, where micro refers to the Planck scale 10^{-33} cm or below, energy scale of 10^{19} GeV or above. Familiar representatives are string theory [10, 11], canonical loop quantum gravity [12, 13, 14, 15, 16, 17], spin network [18, 19, 20], group field theory [19, 21], asymptotic safety [5, 22, 23, 24, 25, 26], simplicies [27, 28, 29, 30, 31], causal dynamical triangulation [32, 33, 34], causal sets [35, 36, 37], etc. We shall comment in the last chapter of this book on how stochastic gravity is linked to quantum gravity, and how it can assist in unraveling the microscopic structures of spacetime.

These nonperturbative theories operative at the Planck scale are what quantum gravity entails. Their structures and contents are fundamentally different from *perturbative* quantum gravity [38, 39] built upon the quantized weak perturbations of classical background spacetimes – the spin-two gravitons, which should exist in nature and are in principle detectable [40, 41] by laboratory experiments at energies much lower than the Planck energy. First explored in the early 1960s [42] in the realm of particle physics [43], graviton physics shares many similar features with photon physics in quantum electrodynamics (QED) and stands on the same footing as the physics of intermediate bosons such as gluons in quantum chromodynamics (QCD) of strong interaction. It is the latter, perturbative quantum gravity, centered on graviton interactions, which falls in the realm of gravitational quantum physics. Therefore, it deals with gravitational effects on quantum systems readily accessible at today's low energy in experiments on Earth or in space. In Sec. 1.3 we shall describe the relation of semiclassical gravity with gravitational quantum physics, point out the non-relation with the Newton–Schrödinger equation [44], and mention the role of stochastic gravity in addressing quantum information issues.

In this section we discuss two key issues, (1) self-consistent backreaction and non-Markovian dynamics; (2) coarse-graining, fluctuations and colored noise. Using an example from semiclassical gravity we point out the necessity of self-consistency in seeking simultaneous solutions of the equations of motion for the quantum matter field and the Einstein equation for the background spacetime, and the importance of including fluctuations in this consideration. In Sec. 1.2 we discuss two main themes: (1) the existence of a stochastic regime in relation to the quantum and the semiclassical regimes; (2) how the conceptual framework of open quantum systems and the influence functional, or its close kin, the 'in-in' or closed-time-path (CTP) or Schwinger–Keldysh effective action, are useful to connect these three levels of theoretical structures and the description of a physical system at each of these three levels. We use the more familiar moving charge quantum field system to illustrate how a correctly formulated approach to self-consistent backreaction leads to a modified Abraham–Lorentz–Dirac (ALD) equation for the motion of a charge with radiation-reaction which is pathology-free, while including the noise from the quantum field we obtain an ALD-Langevin equation describing the stochastic dynamics of the moving charge. In Sec. 1.4 we describe our approach and emphasis, then provide a guide to the readers.

1.1 From QFT in Curved Spacetime to Semiclassical and Stochastic Gravity

We begin with the two solid foundations of modern physics, both having stood the test of time: quantum field theory (QFT) for the description of matter, and general relativity for the description of the large-scale structure and dynamics of spacetime. Placing these two together for the description of quantum matter in a classical gravitational field yields quantum field theory (QFT) in curved spacetime (CST) – let us call this the Level 1 structure. This theory, largely accomplished in the 1970s, is now blessed with many excellent reviews and monographs [45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58]. The next two levels of structure, semiclassical gravity (SCG) at Level 2, established in the 1980s and stochastic gravity (StoG) at Level 3, commenced from the 1990s, are the target of investigation of this book.

At the first structural level, quantum field theory in curved spacetime describes the behavior of a quantum matter field, treated as a test field, propagating in a specified classical background spacetime. Important processes described by this theory range from the *Casimir effect* of quantum fields in spacetimes with boundaries [59, 60, 61, 62] or non-trivial topology [63, 64, 65], to effects of vacuum polarization and vacuum fluctuations such as *particle creation* in the early universe [66, 67, 68, 69, 70, 71, 72, 73, 74, 46, 75, 76], and Hawking radiation of black holes [77, 78, 79, 80, 81, 82, 83], all in the first decade of its development.

The second structural level towards understanding the interaction of quantum fields with gravity is *backreaction*, i.e., the effects of quantum matter fields exerted on the spacetime, impacting on its structure and dynamics. Since the background spacetime remains classical the quantum matter acting as the source would have to come from the expectation value of the stress-energy tensor operator for the quantum fields with respect to some quantum state of symmetries commensurate with the spacetime. Since this object is quadratic in the field operators, which are only well defined as distributions on the spacetime, it contains ultraviolet divergences. Finding viable ways to regularize or renormalize this quantity defined the task of the second stage, in the mid-70s, in the theoretical development of QFTCST. Major regularization methods include adiabatic [84, 85, 86, 87], or 'n-wave' [69, 71, 88] regularization of quantum fields in dynamical spacetimes, suitable for cosmological particle creation processes, dimensional regularization [89, 90, 91, 92] which was successfully applied earlier to proving the renormalizability of QCD [93, 94, 95], the elegant zeta function method [96, 97, 98, 99] for quantum fields in spacetimes with Euclidean sections and the covariant point-separation method [46, 100, 101]. This period ended in 1978, when different regularization methods converge in producing (almost) the same results. The essential uniqueness (modulo some terms quadratic in the spacetime curvature which are independent of the quantum state) in the expectation value of the stress-energy operator via reasonable regularization techniques was proved by Wald [102, 103]. The criteria that a physically meaningful expectation value of the stress-energy tensor ought to satisfy are known as Wald's axioms.

The theory obtained from a self-consistent solution of the geometry and dynamics of the spacetime and the quantum matter field is known as *semi-classical gravity*. Determining the dynamics of spacetime with self-consistent backreaction of the quantum matter field is thus its central task: one assumes a general class of spacetime where the quantum fields live in and act on, and seeks solutions which satisfy simultaneously the Einstein equation for the spacetime and the field equations for the quantum fields. The Einstein equation which has the expectation value of the stress-energy operator of the quantum matter field as the source is known as the *semiclassical Einstein equation*. Semiclassical gravity was first investigated in cosmological backreaction problems in [104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114]. An example is the damping of anisotropy in Bianchi universes by the backreaction of particles created from the vacuum. Using the effect of quantum field processes such as particle creation to explain why the universe became isotropic in the context of chaotic cosmology [115, 116, 117] was investigated in the late seventies.

A well-known example of semiclassical gravity is the inflationary cosmology proposed in the early eighties by Guth [118] and others [119, 120, 121, 122, 342] where the vacuum expectation value of a gauge or Higgs field acts as source in the Einstein equation. It is easy to see that a constant energy density in a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe gives rise to exponential expansion, the case of eternal inflation described by a de Sitter universe. Such a solution is disallowed in classical cosmology because it corresponds to an unphysical equation of state where the pressure $p = -\rho$ energy density. A quantum source makes this solution of the semiclassical Einstein equation not only possible, but, as later development of inflationary cosmology showed, desirable.

Extending semiclassical gravity to *stochastic semiclassical gravity* is a Level 3 theoretical structure developed in the nineties. While semiclassical gravity is based on the semiclassical Einstein equation with the source given by the expectation value of the stress-energy tensor of quantum fields, stochastic semiclassical gravity includes also its fluctuations in a new stochastic semiclassical Einstein–Langevin equation. We will often use the shortened term *stochastic gravity* as there is no confusion as to the nature and source of stochasticity in gravity, here being induced by the quantum matter fields and not from classical sources (e.g., Moffett's theory [123]) or residing *ab initio* in the classical spacetime.

If the centerpiece of semiclassical gravity is the vacuum expectation value of the stress-energy tensor of a quantum field, the centerpiece in stochastic semiclassical gravity is the symmetrized stress-energy bitensor and its expectation value known as the *noise kernel*. The mathematical properties of this quantity, its physical contents in relation to the behavior of fluctuations of quantum fields in curved spacetimes and their backreaction in the spacetime dynamics engendering induced metric fluctuations are the main focus of this theory. How the noises associated with the fluctuations of quantum matter fields seed the structures of the universe, how they affect fluctuations of the black hole horizon and the backreaction of Hawking radiation on the black hole dynamics, as well as the implications on trans-Planckian physics, are new horizons to explore. With regard to the theoretical issues, stochastic gravity is the necessary foundation to investigate the validity of semiclassical gravity. It is also a useful platform supported by well-established low energy (sub-Planckian) physics to explore the connection with high energy (Planckian) physics in the realm of quantum gravity.

If the main issues in QFTCST are finding physically meaningful definitions of different vacua and how the divergences in the expectation values of the stress-energy tensor can be controlled by regularization and renormalization, then the main issues of semiclassical gravity are the self-consistent backreaction of matter fields and their effects on the structure and dynamics of spacetime. For stochastic gravity, the main issues are coarse-graining, noise and fluctuations. Let us select a sample problem (with details extracted from a later chapter) to illustrate the qualitative features of a backreaction problem in semiclassical gravity.

1.1.1 Self-Consistent Backreaction: Nonlocal Dissipation in Open Systems

Backreaction of quantum processes refers to the effects these processes have on the source which engenders them. The source could be a strong or time-dependent background field and the process may entail vacuum polarization or amplified fluctuation effects manifesting as particle creation. A well known classic example is the Schwinger process [124] of particle production in strong static fields. Particle pairs back-reacting on the field engenders dissipative dynamics, which tends to weaken the field [125, 126, 127]. For time-dependent fields, an active topic of current research is the dynamical Casimir effect, where vacuum fluctuations are parametrically amplified into real particles. Indeed, the mechanism is the same as in cosmological particle creation, and the backreaction problem is of interest there because it can significantly alter the dynamics of the early universe near the Planck time. In like manner particle creation from a moving mirror is an analog of Hawking radiation from black holes (or Unruh radiation from an accelerated detector), though the physics is different from cosmological particle creation, as spacetimes in this class (including the de Sitter universe) possess event horizons and share the same characteristic thermal distribution of particles created as a result of exponential red-shifting of the wave modes between the 'in' and 'out' states. The backreaction of Hawking radiation is of interest because it can alter the fate of an evaporating black hole emitting radiation and impacts on the related issues of black hole end state and information loss.

In semiclassical gravity one considers the effects of quantum matter field processes such as vacuum polarization (e.g., trace anomaly) and vacuum fluctuation (e.g., particle creation) exerted on the classical background spacetime. At the equation of motion level, the backreaction problem entails solving in a self-consistent manner both the matter field equations and the Einstein equations with the expectation values of the matter field as source (e.g., [107, 108]). Alternatively one can take the functional approach by integrating over the radiative contributions of the quantum matter field to obtain a one-loop effective action of the background gravitational field. From the variation of this effective action one obtains the equation of motion for the background spacetime now with the backreaction of matter field incorporated therein.

Two aspects in a backreaction problem at the semiclassical level stand out: one, related to backreaction, is the importance of self-consistency in the semiclassical Einstein equation; the other, related to semiclassicality, is decoherence in the quantum to classical transition and the appearance of dissipative dynamics.

1. Self-consistency in semiclassical gravity The necessity of self-consistency in the semiclassical backreaction problem is shown by Flanagan and Wald [128] who used the averaged null energy condition (ANEC) as a criterion to verify this requirement. The expectation value $\langle T_{ab} \rangle$ of the renormalized stress-energy tensor of quantum fields generically violates the classical, local positive energy conditions of general relativity. Nevertheless, it is possible that $\langle T_{ab} \rangle$ may still satisfy some nonlocal positive energy conditions, ANEC being the most prominent. It states that $\int \langle T_{ab} \rangle k^a k^b \, d\lambda \geq 0$ along any complete null geodesic, where k^a denotes the geodesic tangent, with affine parameter λ. If ANEC holds, then traversable wormholes cannot occur. However, although ANEC holds in Minkowski spacetime, it is known that ANEC can be violated in curved spacetimes if one is allowed to choose the spacetime and quantum state arbitrarily, without imposition of the semiclassical Einstein equation, $G_{ab} = 8\pi G_{\rm N} \langle T_{ab} \rangle$. Flanagan and Wald study a free, linear, massless scalar field with arbitrary curvature coupling in the context of perturbation theory about the flat spacetime/vacuum solution. At first order in the metric and state perturbations, and for pure states of the scalar field, they find that the ANEC integral vanishes, as it must for any positivity result to hold. For mixed states, the ANEC integral can be negative. However, they proved that if the ANEC integral transverse to the geodesic is averaged, using a suitable Planck scale smearing function, a strictly positive result is obtained in all cases except for the trivial flat spacetime/vacuum solution. These results suggest that if traversable wormholes do exist as self-consistent solutions of the semiclassical equations, they can only be Planck size. Their finding is in agreement with conclusions drawn by Ford and Roman [129, 130] from different arguments. (See also [131, 132, 133].)

2. Coarse-graining, decoherence and dissipation The procedure of integrating over fluctuations of the quantum field to obtain an effective action, and from there an effective equation of motion, is a form of coarse-graining, which is arguably the most important element in an open-system way of thinking [134, 135, 136, 137, 138]. There, a closed system C is divided into a system S of interest, in this case, the gravitational field, and its environment E, the quantum matter field. One can actually aim higher, and begin with a closed quantum system made up of a quantum gravity sector and a quantum matter sector. This would have been the case if we had a viable theory of quantum gravity – a theory for the microscopic constituents of spacetime and matter. One can then ask what conditions are necessary for the gravity sector to become classical. This was indeed explored in the early 90s, in the realm of quantum cosmology and semiclassical gravity. A necessary ingredient is decoherence, which can be understood in several ways, such as by way of decoherent or consistent histories [139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164] or via environment-induced decoherence [165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179]. We shall describe this aspect later in brief. The study of semiclassical and stochastic gravity begins with the stage where the gravity sector has already been decohered sufficiently that it can be treated as a classical entity. In the next section we shall illustrate this procedure with the simple example of a particle–quantum field interaction, where we know the quantum theory for both the particle and the field. Here we wish to first highlight the main themes so we can have a better grasp of the key issues in semiclassical and stochastic gravity. In doing so we also hope to provide a good motivation for adopting the open quantum system conceptual framework for deeper inquires.

An Example

Let us examine a typical backreaction problem to highlight the second feature above, in two parts: (i) backreaction in the form of dissipation in the open system; (ii) coarse-graining of the environment, noise and fluctuations. We will extract the results of the calculations and spare the reader of the details, which are to be presented in Chapter 3.

Consider a massless conformally coupled quantum scalar field in 4 dimensions obeying the wave equation (2.31) in a classical radiation-filled Bianchi Type I universe with line element (3.55). (When the anisotropy β_{ij} in (3.55) or Q in (2.31) goes to zero, one recovers the radiation-dominated FLRW universe with scale factor a.) For spatially flat cosmology, only the expansion rate, i.e. the derivatives of a, β_{ij} are physically meaningful. The one-loop effective action incorporates the effects of the quantum field on the background geometry. In the Feynman diagram depiction, the loops account for quantum contributions of matter fields and the external legs attached to the loops represent the classical contributions of the gravitational field. The order of the vertices corresponds to the order of the coupling parameter (in this case β') in the perturbative expansion. The one-loop 'in-out' effective action for this problem was calculated in [110, 111]. But for an initial value problem where the evolutionary history of the system (rather than the transition amplitude) is desired one should use the 'in-in' (Schwinger–Keldysh or closed-time-path) effective action [180, 181] (expounded in Chapter 3) because only it can produce equations of motion which are real and causal [182]. The equation of motion for β_{ij} with backreaction from the created particles incorporated therein is an example of the semiclassical Einstein (SCE) equation.

It is convenient to work with a first integral of the SCE equation (3.107), where J_{ij} is an external source for switching on the anisotropy in the distant past and $c_{ij} = \int J_{ij}(\eta) d\eta$ (over conformal time η related to cosmic time t by $\eta = \int dt/a(\eta)$) is an integration constant which sets the magnitude and orientation of the initial anisotropy.

We can write this equation in a schematic form;

$$\frac{d}{d\eta}\left(\tilde{M}\frac{dq_{ij}}{d\eta}\right) + \mathcal{K}\frac{dq_{ij}}{d\eta} + kq_{ij} = c_{ij}, \tag{1.1}$$

where

$$\tilde{M} = \frac{1}{30(4\pi)^2} ln(\tilde{\mu} a), \tag{1.2}$$

$$k = -\frac{a^2}{8\pi G_{\rm N}} + \frac{1}{90(4\pi)^2}\left[\left(\frac{a'}{a}\right)^2 + \left(\frac{a''}{a}\right)\right], \tag{1.3}$$

$$\mathcal{K} q_{ij} = \int d\eta_2 \int d\eta_1 f(\eta_2 - \eta_1)\frac{dq_{ij}}{d\eta_1}, \tag{1.4}$$

where $\tilde{\mu}$ is the renormalization scale, a prime denotes $d/d\eta$ and $\mathcal{K}$ is a non-local operator acting on the function q_{ij}.

This equation is in the form of a damped driven harmonic oscillator where the "generalized coordinates" q_{ij} are the rate of anisotropic expansion β'_{ij} and the "generalized driving force" is c_{ij}. The "spring consant" k is time-dependent, so is the real mass $\tilde{M}$ (strictly speaking the damped harmonic oscillator analogy applies only when these quantities are positive), and the viscous force is velocity-dependent. The non-local kernel $\mathcal{K}$ links the "velocity" q_{ij} at different times, giving rise to a viscosity function γ encapsulating the effects of particle creation which is history-dependent. This memory effect reflects the non-Markovian nature of the resultant semiclassical geometrodynamics, and as we shall see, is a rather generic feature of backreaction processes. This is easy to understand from an open system viewpoint, the time scale of the natural dynamics of the system is different from that of the environment. When one incorporates the dynamics of the environment into that of the system – mathematically it entails turning two ordinary differential equations, one for each party, into an integro-differential equation for just one of them – the dynamics of the open system now contains two time scales reflected in the nonlocal kernel of the integro-differential equation [134].

With a real and causal equation of motion for $q_{ij} = \beta'_{ij}$ one can take the Fourier transform and identify from the dissipative term $i\omega\gamma q(\omega)$ (where $q \equiv q_{ij}q^{ij}$), i.e. the "resistance" component in a LCR circuit, the viscosity function $\gamma(\omega)$:

$$\gamma(\omega) = \frac{|\omega|^3}{60(4\pi)^2}. \tag{1.5}$$

The damping of anisotropy going like ω^4 translates to a dependence on the quadrature of the second derivative of β_{ij}, which can be identified as the lowest order terms of the Weyl curvature tensor. This leads to the result that the rate of particle production in anisotropic or inhomogeneous cosmological spacetimes is proportional to the Weyl curvature-squared $C_{abcd}C^{abcd}$ of the background geometry.

To check if it is correct to associate this viscosity function for the damping of anisotropy of spacetime with particle creation of the scalar field, one can calculate the energy dissipated in the spacetime dynamics within the history of the universe and the total energy of particles created in the process. If one chooses to look at the geometrodynamics (the left-hand side of the SCE equation) one can obtain the (spectral) power $P(\omega)$ dissipated by a velocity-dependent viscous force $\mathbf{F}$ acting on the background spacetime simply from $P(\omega) = \mathbf{F} \cdot \mathbf{v}$. The dissipated energy density $\rho(\omega)$ is obtained by integrating this '(spectral) braking power' $P(\omega)$ over all frequencies.

$$\rho_{dissipation} = \int_0^\infty \frac{d\omega}{2\pi} [\omega\beta_{ij}(\omega)^*][\gamma(\omega)\omega\beta_{ij}(\omega)]. \tag{1.6}$$

Alternatively, focusing on the matter field sector (the right-hand side of the SCE equation) one can calculate the energy density of particles created from the vacuum. The power spectrum of particle pairs created by a given anisotropy history is given by

$$\mathcal{P}(\omega) = \frac{1}{30\pi^2}\omega^4 Tr\beta^*(2\omega)\beta(2\omega). \tag{1.7}$$

Integrating over the full spectrum $\int_0^\infty d\omega(2\omega)\mathcal{P}(\omega)$ produces the total energy density of particle pairs created, which is seen to be precisely equal to the energy density dissipated in the dynamics of spacetime.

This example illustrates that particle creation indeed exerts a dissipative effect on the background gravitational field. In particular, we have given a field-theoretical derivation of the viscosity function of the anisotropy damping process. One can perform similar calculations of particle creation of non-conformal fields in isotropic universes and obtain the viscosity function from its backreaction on the background spacetime. The rate of particle production is proportional to the scalar curvature-squared ξR^2 in FLRW spacetimes. For massive particles there will be a delta-function threshold.

This is all very nice, one may say, but it is only half of the story. An additional term of a stochastic nature has a reserved seat on the right hand side of (1.1) but has escaped our attention so far. The identification of it makes up the second part of this story, highlighting the second key issue, that of noise and fluctuations.

1.1.2 Fluctuations: Colored Noise from Coarse-Graining the Environment

The above example brings out three main themes of relevance to the subject matter we shall develop in this book. (1) Surpassing the theoretical structure of a prescribed curved spacetime exerting a one-way influence on the quantum fields living in it, the backreaction problem demands an account of the mutual influence of quantum matter fields present and the background spacetime in a self-consistent manner as embodied in the semiclassical Einstein equation with the expectation value of the stress-energy tensor of quantum fields acting as its source, which defines semiclassical gravity. (2) The backreaction of quantum field processes such as particle creation alters the state and evolution of the background spacetime, generally resulting in the appearance of *nonlocal dissipative dynamics*. The physical meaning of dissipation can be seen more clearly in the conceptual framework of *open quantum systems*: A coarse-grained environment backreacting on a system results in an open system whose dynamics is no longer necessarily unitary. (3) In an open system perspective, where the classical spacetime is viewed as the system and the quantum matter field is viewed as its environment, a coarse-grained environment can under some rather general conditions (Gaussian systems for certain) be represented by *a noise term, usually colored*. This stochastic forcing term represents the fluctuations in the environment variables. The inclusion of the fluctuations of the stress-energy tensor entering as noise turns the semiclassical Einstein equation into an Einstein–Langevin (E–L) equation which defines stochastic gravity. With a noise of zero mean under stochastic average, taking the stochastic average of the E–L eqn reproduces the semiclassical Einstein equation. It is in this sense that the semiclassical gravity is regarded as a mean field theory.

Historically, the development of stochastic gravity took three stages. Stage 1 began around 1977, when different regularization schemes came to agreement enough to facilitate a proper treatment of the backreaction problems. By 1987 this task was largely completed, which led to the establishment of semiclassical gravity. An important step is the realization that the 'in-out' effective action must be replaced by the 'in-in' effective action to ensure a real and causal equation of motion for the spacetime dynamics. Stage 3 began in 1994 when the Einstein–Langevin equation was first proposed, followed by several worked-out examples. Why should there be a stochastic term and why was it not appearing earlier in the semiclassical Einstein equation – these were the questions asked and answered in the intervening years which marked Stage 2. By 1996 the basic elements of stochastic gravity were in place and the theoretical structure largely completed by 2000 [183]. The ensuing years saw applications of stochastic gravity to the structure formation problem in cosmology and the backreaction and fluctuations problems in black holes, as well as continual developments in the formulation of a validity criterion for semiclassical gravity and the calculation of the noise kernels or the stress-energy tensor correlations for spacetimes of importance to cosmology and black hole physics.

Let us see what this entails with the example described above. The statement is that a stochastic term s_{ij} can be accommodated on the right-hand side of (1.1)

$$\frac{d}{d\eta}\left(\tilde{M}\frac{dq_{ij}}{d\eta}\right) + \mathcal{K}\frac{dq_{ij}}{d\eta} + kq_{ij} = c_{ij} + s_{ij}, \tag{1.8}$$

with $s_{ij}(\eta) = \int d\eta' \xi_{ij}(\eta')$ where $\xi_{ij}(\eta)$ is a Gaussian type noise, which is completely characterized by its second moment