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Sheldon M. Ross
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# An Elementary Introduction to Mathematical Finance, Third Edition

This textbook on the basics of option pricing is accessible to readers with limited mathematical training. It is for both professional traders and undergraduates studying the basics of finance. Assuming no prior knowledge of probability, Sheldon M. Ross offers clear, simple explanations of arbitrage, the Black–Scholes option pricing formula, and other topics such as utility functions, optimal portfolio selections, and the capital assets pricing model. Among the many new features of this third edition are new chapters on Brownian motion and geometric Brownian motion, stochastic order relations, and stochastic dynamic programming, along with expanded sets of exercises and references for all the chapters.

Sheldon M. Ross is the Epstein Chair Professor in the Department of Industrial and Systems Engineering, University of Southern California. He received his Ph.D. in statistics from Stanford University in 1968 and was a Professor at the University of California, Berkeley, from 1976 until 2004. He has published more than 100 articles and a variety of textbooks in the areas of statistics and applied probability, including *Topics in Finite and Discrete Mathematics* (2000), *Introduction to Probability and Statistics for Engineers and Scientists, Fourth Edition* (2009), *A First Course in Probability, Eighth Edition* (2009), and *Introduction to Probability Models, Tenth Edition* (2009). Dr. Ross serves as the editor for *Probability in the Engineering and Informational Sciences*.

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# An Elementary Introduction to Mathematical Finance

## Third Edition

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*To my parents,*
Ethel and Louis Ross

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# Introduction and Preface

An *option* gives one the right, but not the obligation, to buy or sell a security under specified terms. A *call option* is one that gives the right to buy, and a *put option* is one that gives the right to sell the security. Both types of options will have an *exercise price* and an *exercise time*. In addition, there are two standard conditions under which options operate: *European* options can be utilized only at the exercise time, whereas *American* options can be utilized at any time up to exercise time. Thus, for instance, a European call option with exercise price $K$ and exercise time $t$ gives its holder the right to purchase at time $t$ one share of the underlying security for the price $K$, whereas an American call option gives its holder the right to make the purchase at any time before or at time $t$.

A prerequisite for a strong market in options is a computationally efficient way of evaluating, at least approximately, their worth; this was accomplished for call options (of either American or European type) by the famous Black–Scholes formula. The formula assumes that prices of the underlying security follow a geometric Brownian motion. This means that if $S(y)$ is the price of the security at time $y$ then, for any price history up to time $y$, the ratio of the price at a specified future time $t + y$ to the price at time $y$ has a lognormal distribution with mean and variance parameters $t\mu$ and $t\sigma^2$, respectively. That is,

$$\log\left(\frac{S(t+y)}{S(y)}\right)$$

will be a normal random variable with mean $t\mu$ and variance $t\sigma^2$. Black and Scholes showed, under the assumption that the prices follow a geometric Brownian motion, that there is a single price for a call option that does not allow an idealized trader – one who can instantaneously make trades without any transaction costs – to follow a strategy that will result in a sure profit in all cases. That is, there will be no certain profit (i.e., no *arbitrage*) if and only if the price of the option is as given by the Black–Scholes formula. In addition, this price depends only on the

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variance parameter $\sigma$ of the geometric Brownian motion (as well as on the prevailing interest rate, the underlying price of the security, and the conditions of the option) and not on the parameter $\mu$. Because the parameter $\sigma$ is a measure of the volatility of the security, it is often called the *volatility* parameter.

A *risk-neutral* investor is one who values an investment solely through the expected present value of its return. If such an investor models a security by a geometric Brownian motion that turns all investments involving buying and selling the security into fair bets, then this investor's valuation of a call option on this security will be precisely as given by the Black–Scholes formula. For this reason, the Black–Scholes valuation is often called a *risk-neutral valuation*.

Our first objective in this book is to derive and explain the Black–Scholes formula. Its derivation, however, requires some knowledge of probability, and this is what the first three chapters are concerned with. Chapter 1 introduces probability and the probability experiment. Random variables – numerical quantities whose values are determined by the outcome of the probability experiment – are discussed, as are the concepts of the expected value and variance of a random variable. In Chapter 2 we introduce normal random variables; these are random variables whose probabilities are determined by a bell-shaped curve. The central limit theorem is presented in this chapter. This theorem, probably the most important theoretical result in probability, states that the sum of a large number of random variables will approximately be a normal random variable. In Chapter 3 we introduce the geometric Brownian motion process; we define it, show how it can be obtained as the limit of simpler processes, and discuss the justification for its use in modeling security prices.

With the probability necessities behind us, the second part of the text begins in Chapter 4 with an introduction to the concept of interest rates and present values. A key concept underlying the Black–Scholes formula is that of arbitrage, which is the subject of Chapter 5. In this chapter we show how arbitrage can be used to determine prices in a variety of situations, including the single-period binomial option model. In Chapter 6 we present the arbitrage theorem and use it to find an expression for the unique nonarbitrage option cost in the multiperiod binomial model. In Chapter 7 we use the results of Chapter 6, along with the approximations of geometric Brownian motion presented in Chapter 4, to obtain a

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simple derivation of the Black–Scholes equation for pricing call options. Properties of the resultant option cost as a function of its parameters are derived, as is the delta hedging replication strategy. Additional results on options are presented in Chapter 8, where we derive option prices for dividend-paying securities; show how to utilize a multiperiod binomial model to determine an approximation of the risk-neutral price of an American put option; determine no-arbitrage costs when the security's price follows a model that superimposes random jumps on a geometric Brownian motion; and present different estimators of the volatility parameter.

In Chapter 9 we note that, in many situations, arbitrage considerations do not result in a unique cost. We show the importance in such cases of the investor's utility function as well as his or her estimates of the probabilities of the possible outcomes of the investment. The concepts of mean variance analysis, value and conditional value at risk, and the capital assets pricing model are introduced.

In Chapter 10 we introduce stochastic order relations. These relations can be useful in determining which of a class of investments is best without completely specifying the investor's utility function. For instance, if the return from one investment is greater than the return from another investment in the sense of first-order stochastic dominance, then the first investment is to be preferred for any increasing utility function; whereas if the first return is greater in the sense of second-order stochastic dominance, then the first investment is to be preferred as long as the utility function is concave and increasing.

In Chapters 11 and 12 we study some optimization models in finance. In Chapter 13 we introduce some nonstandard, or "exotic," options such as barrier, Asian, and lookback options. We explain how to use Monte Carlo simulation, implementing variance reduction techniques, to efficiently determine their geometric Brownian motion risk-neutral valuations.

The Black–Scholes formula is useful even if one has doubts about the validity of the underlying geometric Brownian model. For as long as one accepts that this model is at least approximately valid, its use gives one an idea about the *appropriate* price of the option. Thus, if the actual trading option price is below the formula price then it would seem that the option is underpriced in relation to the security itself, thus leading one to consider a strategy of buying options and selling the security

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(with the reverse being suggested when the trading option price is above the formula price). In Chapter 14 we show that real data cannot aways be fit by a geometric Brownian motion model, and that more general models may need to be considered. In the case of commodity prices, there is a strong belief by many traders in the concept of mean price reversion: that the market prices of certain commodities have tendencies to revert to fixed values. In Chapter 15 we present a model, more general than geometric Brownian motion, that can be used to model the price flow of such a commodity.

## *New to This Edition*

Whereas the third edition contains changes in almost all previous chapters, the major changes in the new edition are as follows.

- Chapter 3 on *Brownian Motion and Geometric Brownian Motion* has been completely rewritten. Among other things the new chapter gives an elementary derivation of the distribution of the maximum variable of a Brownian motion process with drift, as well as an elementary proof of the Cameron–Martin theorem.
- Section 7.5.2 has been reworked, clarifying the argument leading to a simple derivation of the partial derivatives of the Black–Scholes call option pricing formula.
- Section 7.6 on *European Put Options* is new. It presents monotonicity and convexity results concerning the risk-neutral price of a European put option.
- Chapter 10 on *Stochastic Order Relations* is new. This chapter presents first- and second-order stochastic dominance, as well as likelihood ratio orderings. Among other things, it is shown (in Section 10.5.1) that a normal random variable decreases, in the second-order stochastic dominance sense, as its variance increases.
- The old Chapter 10 is now Chapter 11.
- Chapter 12 on *Stochastic Dynamic Programming* is new.
- The old Chapter 11 is now Chapter 13. New within this chapter is Section 13.9, which presents continuous time approximations of barrier and lookback options.
- The old Chapter 12 is now Chapter 14.
- The old Chapter 13 is now Chapter 15.

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One technical point that should be mentioned is that we use the notation $\log(x)$ to represent the natural logarithm of $x$. That is, the logarithm has base $e$, where $e$ is defined by

$$e = \lim_{n \to \infty} (1 + 1/n)^n$$

and is approximately given by $2.71828\ldots$.

We would like to thank Professors Ilan Adler and Shmuel Oren for some enlightening conversations, Mr. Kyle Lin for his many useful comments, and Mr. Nahoya Takezawa for his general comments and for doing the numerical work needed in the final chapters. We would also like to thank Professors Anthony Quas, Daniel Naiman, and Agostino Capponi for helpful comments concerning the previous edition.

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