Cambridge University Press
978-0-521-19253-8 - An Elementary Introduction to Mathematical Finance, Third Edition
Sheldon M. Ross
Excerpt
More information

1. Probability

1.1 Probabilities and Events

Consider an experiment and let S, called the *sample space,* be the set of all possible outcomes of the experiment. If there are m possible outcomes of the experiment then we will generally number them 1 through m, and so $S = \{1, 2, \ldots, m\}$. However, when dealing with specific examples, we will usually give more descriptive names to the outcomes.

Example 1.1a (i) Let the experiment consist of flipping a coin, and let the outcome be the side that lands face up. Thus, the sample space of this experiment is

$$S = \{h, t\},$$

where the outcome is h if the coin shows heads and t if it shows tails.

(ii) If the experiment consists of rolling a pair of dice – with the outcome being the pair (i, j), where i is the value that appears on the first die and j the value on the second – then the sample space consists of the following 36 outcomes:

$$\begin{array}{cccccc}
(1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\
(2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\
(3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\
(4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\
(5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\
(6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6).
\end{array}$$

(iii) If the experiment consists of a race of r horses numbered 1, 2, 3, $\ldots, r$, and the outcome is the order of finish of these horses, then the sample space is

$$S = \{\text{all orderings of the numbers } 1, 2, 3, \ldots, r\}.$$

Cambridge University Press
978-0-521-19253-8 - An Elementary Introduction to Mathematical Finance, Third Edition
Sheldon M. Ross
Excerpt
More information

For instance, if $r = 4$ then the outcome is $(1, 4, 2, 3)$ if the number 1 horse comes in first, number 4 comes in second, number 2 comes in third, and number 3 comes in fourth. □

Consider once again an experiment with the sample space $S = \{1, 2, \ldots, m\}$. We will now suppose that there are numbers $p_1, \ldots, p_m$ with

$$p_i \geq 0, \; i = 1, \ldots, m, \quad \text{and} \quad \sum_{i=1}^{m} p_i = 1$$

and such that p_i is the *probability* that i is the outcome of the experiment.

Example 1.1b In Example 1.1a(i), the coin is said to be *fair* or *unbiased* if it is equally likely to land on heads as on tails. Thus, for a fair coin we would have that

$$p_h = p_t = 1/2.$$

If the coin were biased and heads were twice as likely to appear as tails, then we would have

$$p_h = 2/3, \qquad p_t = 1/3.$$

If an unbiased pair of dice were rolled in Example 1.1a(ii), then all possible outcomes would be equally likely and so

$$p_{(i,j)} = 1/36, \quad 1 \leq i \leq 6, \; 1 \leq j \leq 6.$$

If $r = 3$ in Example 1.1a(iii), then we suppose that we are given the six nonnegative numbers that sum to 1:

$$p_{1,2,3}, \; p_{1,3,2}, \; p_{2,1,3}, \; p_{2,3,1}, \; p_{3,1,2}, \; p_{3,2,1},$$

where $p_{i,j,k}$ represents the probability that horse i comes in first, horse j second, and horse k third. □

Any set of possible outcomes of the experiment is called an *event*. That is, an event is a subset of S, the set of all possible outcomes. For any event A, we say that A *occurs* whenever the outcome of the experiment is a point in A. If we let $P(A)$ denote the probability that event A occurs, then we can determine it by using the equation

$$P(A) = \sum_{i \in A} p_i. \tag{1.1}$$

Note that this implies

$$P(S) = \sum_i p_i = 1. \tag{1.2}$$

In words, the probability that the outcome of the experiment is in the sample space is equal to 1 – which, since S consists of all possible outcomes of the experiment, is the desired result.

Example 1.1c Suppose the experiment consists of rolling a pair of fair dice. If A is the event that the sum of the dice is equal to 7, then

$$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

and

$$P(A) = 6/36 = 1/6.$$

If we let B be the event that the sum is 8, then

$$P(B) = p_{(2,6)} + p_{(3,5)} + p_{(4,4)} + p_{(5,3)} + p_{(6,2)} = 5/36.$$

If, in a horse race between three horses, we let A denote the event that horse number 1 wins, then $A = \{(1, 2, 3), (1, 3, 2)\}$ and

$$P(A) = p_{1,2,3} + p_{1,3,2}. \qquad \square$$

For any event A, we let A^c, called the *complement* of A, be the event containing all those outcomes in S that are not in A. That is, A^c occurs if and only if A does not. Since

$$\begin{aligned} 1 &= \sum_i p_i \\ &= \sum_{i \in A} p_i + \sum_{i \in A^c} p_i \\ &= P(A) + P(A^c), \end{aligned}$$

we see that

$$P(A^c) = 1 - P(A). \tag{1.3}$$

That is, the probability that the outcome is not in A is 1 minus the probability that it is in A. The complement of the sample space S is the null event $\emptyset$, which contains no outcomes. Since $\emptyset = S^c$, we obtain from

Equations (1.2) and (1.3) that

$$P(\emptyset) = 0.$$

For any events A and B we define $A \cup B$, called the *union* of A and B, as the event consisting of all outcomes that are in A, or in B, or in both A and B. Also, we define their *intersection* AB (sometimes written $A \cap B$) as the event consisting of all outcomes that are both in A and in B.

Example 1.1d Let the experiment consist of rolling a pair of dice. If A is the event that the sum is 10 and B is the event that both dice land on even numbers greater than 3, then

$$A = \{(4, 6), (5, 5), (6, 4)\}, \qquad B = \{(4, 4), (4, 6), (6, 4), (6, 6)\}.$$

Therefore,

$$A \cup B = \{(4, 4), (4, 6), (5, 5), (6, 4), (6, 6)\},$$
$$AB = \{(4, 6), (6, 4)\}. \qquad \square$$

For any events A and B, we can write

$$P(A \cup B) = \sum_{i \in A \cup B} p_i,$$
$$P(A) = \sum_{i \in A} p_i,$$
$$P(B) = \sum_{i \in B} p_i.$$

Since every outcome in both A and B is counted twice in $P(A) + P(B)$ and only once in $P(A \cup B)$, we obtain the following result, often called the *addition theorem of probability.*

Proposition 1.1.1

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

Thus, the probability that the outcome of the experiment is either in A or in B equals the probability that it is in A, plus the probability that it is in B, minus the probability that it is in both A and B.

Example 1.1e Suppose the probabilities that the Dow-Jones stock index increases today is .54, that it increases tomorrow is .54, and that it increases both days is .28. What is the probability that it does not increase on either day?

Solution. Let A be the event that the index increases today, and let B be the event that it increases tomorrow. Then the probability that it increases on at least one of these days is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) \\ &= .54 + .54 - .28 = .80. \end{aligned}$$

Therefore, the probability that it increases on neither day is $1 - .80 = .20$. □

If $AB = \emptyset$, we say that A and B are *mutually exclusive* or *disjoint*. That is, events are mutually exclusive if they cannot both occur. Since $P(\emptyset) = 0$, it follows from Proposition 1.1.1 that, when A and B are mutually exclusive,

$$P(A \cup B) = P(A) + P(B).$$

1.2 Conditional Probability

Suppose that each of two teams is to produce an item, and that the two items produced will be rated as either acceptable or unacceptable. The sample space of this experiment will then consist of the following four outcomes:

$$S = \{(a, a), (a, u), (u, a), (u, u)\},$$

where (a, u) means, for instance, that the first team produced an acceptable item and the second team an unacceptable one. Suppose that the probabilities of these outcomes are as follows:

$$\begin{aligned} P(a, a) &= .54, \\ P(a, u) &= .28, \\ P(u, a) &= .14, \\ P(u, u) &= .04. \end{aligned}$$

If we are given the information that exactly one of the items produced was acceptable, what is the probability that it was the one produced by the first team? To determine this probability, consider the following reasoning. Given that there was exactly one acceptable item produced, it follows that the outcome of the experiment was either (a, u) or (u, a). Since the outcome (a, u) was initially twice as likely as the outcome (u, a), it should remain twice as likely given the information that one of them occurred. Therefore, the probability that the outcome was (a, u) is $2/3$, whereas the probability that it was (u, a) is $1/3$.

Let $A = \{(a, u), (a, a)\}$ denote the event that the item produced by the first team is acceptable, and let $B = \{(a, u), (u, a)\}$ be the event that exactly one of the produced items is acceptable. The probability that the item produced by the first team was acceptable given that exactly one of the produced items was acceptable is called the *conditional probability* of A given that B has occurred; this is denoted as

$$P(A|B).$$

A general formula for $P(A|B)$ is obtained by an argument similar to the one given in the preceding. Namely, if the event B occurs then, in order for the event A to occur, it is necessary that the occurrence be a point in both A and B; that is, it must be in AB. Now, since we know that B has occurred, it follows that B can be thought of as the new sample space, and hence the probability that the event AB occurs will equal the probability of AB relative to the probability of B. That is,

$$P(A|B) = \frac{P(AB)}{P(B)}. \tag{1.4}$$

Example 1.2a A coin is flipped twice. Assuming that all four points in the sample space $S = \{(h, h), (h, t), (t, h), (t, t)\}$ are equally likely, what is the conditional probability that both flips land on heads, given that

(a) the first flip lands on heads, and
(b) at least one of the flips lands on heads?

Solution. Let $A = \{(h, h)\}$ be the event that both flips land on heads; let $B = \{(h, h), (h, t)\}$ be the event that the first flip lands on heads; and let $C = \{(h, h), (h, t), (t, h)\}$ be the event that at least one of the flips lands on heads. We have the following solutions:

$$\begin{aligned} P(A|B) &= \frac{P(AB)}{P(B)} \\ &= \frac{P(\{(h,h)\})}{P(\{(h,h),(h,t)\})} \\ &= \frac{1/4}{2/4} \\ &= 1/2 \end{aligned}$$

and

$$\begin{aligned} P(A|C) &= \frac{P(AC)}{P(C)} \\ &= \frac{P(\{(h,h)\})}{P(\{(h,h),(h,t),(t,h)\})} \\ &= \frac{1/4}{3/4} \\ &= 1/3. \end{aligned}$$

Many people are initially surprised that the answers to parts (a) and (b) are not identical. To understand why the answers are different, note first that – conditional on the first flip landing on heads – the second one is still equally likely to land on either heads or tails, and so the probability in part (a) is $1/2$. On the other hand, knowing that at least one of the flips lands on heads is equivalent to knowing that the outcome is not (t,t). Thus, given that at least one of the flips lands on heads, there remain three equally likely possibilities, namely (h,h), (h,t), (t,h), showing that the answer to part (b) is $1/3$. □

It follows from Equation (1.4) that

$$P(AB) = P(B)P(A|B). \tag{1.5}$$

That is, the probability that both A and B occur is the probability that B occurs multiplied by the conditional probability that A occurs given that B occurred; this result is often called the *multiplication theorem of probability*.

Example 1.2b Suppose that two balls are to be withdrawn, without replacement, from an urn that contains 9 blue and 7 yellow balls. If each ball drawn is equally likely to be any of the balls in the urn at the time, what is the probability that both balls are blue?

Solution. Let B_1 and B_2 denote, respectively, the events that the first and second balls withdrawn are blue. Now, given that the first ball withdrawn is blue, the second ball is equally likely to be any of the remaining 15 balls, of which 8 are blue. Therefore, $P(B_2|B_1) = 8/15$. As $P(B_1) = 9/16$, we see that

$$P(B_1B_2) = \frac{9}{16}\frac{8}{15} = \frac{3}{10}. \qquad \square$$

The conditional probability of A given that B has occurred is not generally equal to the unconditional probability of A. In other words, knowing that the outcome of the experment is an element of B generally changes the probability that it is an element of A. (What if A and B are mutually exclusive?) In the special case where $P(A|B)$ is equal to $P(A)$, we say that A is *independent* of B. Since

$$P(A|B) = \frac{P(AB)}{P(B)},$$

we see that A is independent of B if

$$P(AB) = P(A)P(B). \tag{1.6}$$

The relation in (1.6) is symmetric in A and B. Thus it follows that, whenever A is independent of B, B is also independent of A – that is, A and B are *independent events.*

Example 1.2c Suppose that, with probability .52, the closing price of a stock is at least as high as the close on the previous day, and that the results for succesive days are independent. Find the probability that the closing price goes down in each of the next four days, but not on the following day.

Solution. Let A_i be the event that the closing price goes down on day i. Then, by independence, we have

$$\begin{aligned} P(A_1A_2A_3A_4A_5^c) &= P(A_1)P(A_2)P(A_3)P(A_4)P(A_5^c) \\ &= (.48)^4(.52) = .0276. \qquad \square \end{aligned}$$

1.3 Random Variables and Expected Values

Numerical quantities whose values are determined by the outcome of the experiment are known as *random variables*. For instance, the sum obtained when rolling dice, or the number of heads that result in a series of coin flips, are random variables. Since the value of a random variable is determined by the outcome of the experiment, we can assign probabilities to each of its possible values.

Example 1.3a Let the random variable X denote the sum when a pair of fair dice are rolled. The possible values of X are $2, 3, \ldots, 12$, and they have the following probabilities:

$$\begin{aligned}
P\{X=2\} &= P\{(1,1)\} = 1/36,\\
P\{X=3\} &= P\{(1,2),(2,1)\} = 2/36,\\
P\{X=4\} &= P\{(1,3),(2,2),(3,1)\} = 3/36,\\
P\{X=5\} &= P\{(1,4),(2,3),(3,2),(4,1)\} = 4/36,\\
P\{X=6\} &= P\{(1,5),(2,4),(3,3),(4,2),(5,1)\} = 5/36,\\
P\{X=7\} &= P\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} = 6/36,\\
P\{X=8\} &= P\{(2,6),(3,5),(4,4),(5,3),(6,2)\} = 5/36,\\
P\{X=9\} &= P\{(3,6),(4,5),(5,4),(6,3)\} = 4/36,\\
P\{X=10\} &= P\{(4,6),(5,5),(6,4)\} = 3/36,\\
P\{X=11\} &= P\{(5,6),(6,5)\} = 2/36,\\
P\{X=12\} &= P\{(6,6)\} = 1/36.
\end{aligned}$$ □

If X is a random variable whose possible values are $x_1, x_2, \ldots, x_n$, then the set of probabilities $P\{X = x_j\}$ $(j = 1, \ldots, n)$ is called the *probability distribution* of the random variable. Since X must assume one of these values, it follows that

$$\sum_{j=1}^{n} P\{X = x_j\} = 1.$$

Definition If X is a random variable whose possible values are $x_1, x_2, \ldots, x_n$, then the *expected value* of X, denoted by $E[X]$, is defined by

$$E[X] = \sum_{j=1}^{n} x_j P\{X = x_j\}.$$

Alternative names for $E[X]$ are the *expectation* or the *mean* of X.

In words, $E[X]$ is a weighted average of the possible values of X, where the weight given to a value is equal to the probability that X assumes that value.

Example 1.3b Let the random variable X denote the amount that we win when we make a certain bet. Find $E[X]$ if there is a 60% chance that we lose 1, a 20% chance that we win 1, and a 20% chance that we win 2.

Solution.

$$E[X] = -1(.6) + 1(.2) + 2(.2) = 0.$$

Thus, the expected amount that is won on this bet is equal to 0. A bet whose expected winnings is equal to 0 is called a *fair* bet. □

Example 1.3c A random variable X, which is equal to 1 with probability p and to 0 with probability $1 - p$, is said to be a *Bernoulli* random variable with parameter p. Its expected value is

$$E[X] = 1(p) + 0(1 - p) = p. \qquad \square$$

A useful and easily established result is that, for constants a and b,

$$E[aX + b] = aE[X] + b. \tag{1.7}$$

To verify Equation (1.7), let $Y = aX + b$. Since Y will equal $ax_j + b$ when $X = x_j$, it follows that

$$\begin{aligned}
E[Y] &= \sum_{j=1}^{n} (ax_j + b) P\{X = x_j\} \\
&= \sum_{j=1}^{n} ax_j P\{X = x_j\} + \sum_{j=1}^{n} bP\{X = x_j\} \\
&= a \sum_{j=1}^{n} x_j P\{X = x_j\} + b \sum_{j=1}^{n} P\{X = x_j\} \\
&= aE[X] + b.
\end{aligned}$$