Cambridge University Press & Assessment
978-0-521-19243-9 — Locally Convex Spaces over Non-Archimedean Valued Fields
C. Perez-Garcia , W. H. Schikhof
Frontmatter
More Information

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 119

LOCALLY CONVEX SPACES OVER NON-ARCHIMEDEAN VALUED FIELDS

Non-Archimedean functional analysis, where alternative but equally valid number systems such as p-adic numbers are fundamental, is a fast-growing discipline widely used not just within pure mathematics, but also applied in other sciences, including physics, biology and chemistry. This book is the first to provide a comprehensive treatment of non-Archimedean locally convex spaces.

The authors provide a clear exposition of the basic theory, together with complete proofs and new results from the latest research. A guide to the many illustrative examples provided, end-of-chapter notes and glossary of terms all make this book easily accessible to beginners at the graduate level, as well as specialists from a variety of disciplines.

C. Perez-Garcia is a Professor in the Department of Mathematics, Statistics and Computation at the University of Cantabria, Spain.

W. H. Schikhof worked as a Professor at Radboud University Nijmegen, the Netherlands for 40 years.

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit: http://www.cambridge.org/series/sSeries.asp?code=CSAM

Already published

66 S. Morosawa *et al. Holomorphic dynamics*
67 A. J. Berrick & M. E. Keating *Categories and modules with K-theory in view*
68 K. Sato *Lévy processes and infinitely divisible distributions*
69 H. Hida *Modular forms and Galois cohomology*
70 R. Iorio & V. Iorio *Fourier analysis and partial differential equations*
71 R. Blei *Analysis in integer and fractional dimensions*
72 F. Borceux & G. Janelidze *Galois theories*
73 B. Bollobás *Random graphs (2nd Edition)*
74 R. M. Dudley *Real analysis and probability (2nd Edition)*
75 T. Sheil-Small *Complex polynomials*
76 C. Voisin *Hodge theory and complex algebraic geometry, I*
77 C. Voisin *Hodge theory and complex algebraic geometry, II*
78 V. Paulsen *Completely bounded maps and operator algebras*
79 F. Gesztesy & H. Holden *Soliton equations and their algebro-geometric solutions, I*
81 S. Mukai *An introduction to invariants and moduli*
82 G. Tourlakis *Lectures in logic and set theory, I*
83 G. Tourlakis *Lectures in logic and set theory, II*
84 R. A. Bailey *Association schemes*
85 J. Carlson, S. Müller-Stach & C. Peters *Period mappings and period domains*
86 J. J. Duistermaat & J. A. C. Kolk *Multidimensional real analysis, I*
87 J. J. Duistermaat & J. A. C. Kolk *Multidimensional real analysis, II*
89 M. C. Golumbic & A. N. Trenk *Tolerance graphs*
90 L. H. Harper *Global methods for combinatorial isoperimetric problems*
91 I. Moerdijk & J. Mrčun *Introduction to foliations and Lie groupoids*
92 J. Kollár, K. E. Smith & A. Corti *Rational and nearly rational varieties*
93 D. Applebaum *Lévy processes and stochastic calculus (1st Edition)*
94 B. Conrad *Modular forms and the Ramanujan conjecture*
95 M. Schechter *An introduction to nonlinear analysis*
96 R. Carter *Lie algebras of finite and affine type*
97 H. L. Montgomery & R. C. Vaughan *Multiplicative number theory, I*
98 I. Chavel *Riemannian geometry (2nd Edition)*
99 D. Goldfeld *Automorphic forms and L-functions for the group* $GL(n, R)$
100 M. B. Marcus & J. Rosen *Markov processes, Gaussian processes, and local times*
101 P. Gille & T. Szamuely *Central simple algebras and Galois cohomology*
102 J. Bertoin *Random fragmentation and coagulation processes*
103 E. Frenkel *Langlands correspondence for loop groups*
104 A. Ambrosetti & A. Malchiodi *Nonlinear analysis and semilinear elliptic problems*
105 T. Tao & V. H. Vu *Additive combinatorics*
106 E. B. Davies *Linear operators and their spectra*
107 K. Kodaira *Complex analysis*
108 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli *Harmonic analysis on finite groups*
109 H. Geiges *An introduction to contact topology*
110 J. Faraut *Analysis on Lie groups: An Introduction*
111 E. Park *Complex topological K-theory*
112 D. W. Stroock *Partial differential equations for probabilists*
113 A. Kirillov, Jr *An introduction to Lie groups and Lie algebras*
114 F. Gesztesy *et al. Soliton equations and their algebro-geometric solutions, II*
115 E. de Faria & W. de Melo *Mathematical tools for one-dimensional dynamics*
116 D. Applebaum *Lévy processes and stochastic calculus (2nd Edition)*
117 T. Szamuely *Galois groups and fundamental groups*
118 G. W. Anderson, A. Guionnet & O. Zeitouni *An introduction to random matrices*

Locally Convex Spaces over Non-Archimedean Valued Fields

C. PEREZ-GARCIA
W. H. SCHIKHOF

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521192439

First published 2010

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-19243-9 Hardback

To my family and to my friend Nicole

-C.P.G.

Contents

Cambridge University Press & Assessment
978-0-521-19243-9 — Locally Convex Spaces over Non-Archimedean Valued Fields
C. Perez-Garcia , W. H. Schikhof
Frontmatter
More Information

Preface

Aim

This book presents the basics of locally convex theory over a field K with a non-Archimedean valuation $|\,.\,| : K \longrightarrow [0, \infty)$ (see 1.2.3).[1] The most important example of such a K is the field of the p-adic numbers (1.2.7). The strong triangle inequality $|\lambda + \mu| \leq \max(|\lambda|\,, |\mu|)$ is the major difference between $|\,.\,|$ and the absolute value function on the field of real numbers $\mathbb{R}$ and the field of complex numbers $\mathbb{C}$. Likewise, the defining seminorms of our locally convex spaces will satisfy the strong triangle inequality.

The book is self-contained in the sense that it does not require knowledge of any deep theory; only basic knowledge of (linear) algebra, analysis and topology are needed. It is intended for both (graduate) students and interested researchers in other areas, but is also of relevance for specialists.

History

The founding father of non-Archimedean Functional Analysis was Monna, who wrote a series of papers in 1943 (see [152]–[155]). Over the years a well-established discipline developed, reflected in the 2000 Mathematics Subject Classifications 46S10 and 47S10 of the Mathematical Reviews. A milestone was reached in 1978 at the publication of van Rooij's book [193], the most extensive treatment on non-Archimedean Banach spaces existing in the literature. In the meantime van Tiel had published his thesis [227] on non-Archimedean locally convex spaces. Both fundamental works still form a basis for new developments, and have been cited by many authors. We should also mention

[1] Note that in this book all proofs, definitions, theorems, etc. are numbered decimally by chapter. They will be referred to by number only.

the proceedings of conferences on non-Archimedean analysis which were held every two years from 1990 onwards, [21], [88], [20], [209], [107], [135], [208], [45], [2], containing several publications on locally convex theory (we have listed the references in chronological order).

The time had come for a volume on locally convex theory to appear and the present book hopefully provides an answer to this need. Our aim is to cover the fundamentals by setting up a general theory which allows a wide spectrum of subjects and examples. Our proof techniques are analytically oriented. In this context we would like to point out Schneider's book [210]. Whereas our book is directed towards a rather general readership, Schneider, as he explains himself, had a different motivation, i.e., to offer a quick grasp to a reader working in other areas (such as number theory). Because of this, he allows for restrictions, for example working mainly over spherically complete fields. Also, the treatment in [210] has a more algebraic flavour. Despite these differences, Schneider's book and ours are compatible and one can be used to complement the reading of the other.

Foreign affairs

Complex numbers provide an excellent domain for forcing quadratic equations to have solutions. Likewise, the p-adic number field acts as a natural home for solving certain infinite systems of congruences, so it takes a prominent place in number theory. But also in other branches (such as algebraic geometry; representation of (Lie) groups; (several) complex variables; real analysis; even theoretical physics, see e.g. [34], [47] and [140]), one is more interested in the role non-Archimedean fields could play as fundamental objects. Researchers in those disciplines sometimes need a solid background in non-Archimedean locally convex spaces; we have mentioned Schneider's reason for writing his book. This has also influenced the set up of our book, as we will later explain.

The presence of the vast area of Functional Analysis over $\mathbb{R}$ or $\mathbb{C}$, henceforth called "classical analysis" is, of course, a fruitful source of inspiration for the non-Archimedean case. However, the reader will notice that, in the course of the development of the theory, the non-Archimedean world is equally fascinating, and asks for a mathematical intuition of its own.

Book organization

In Chapter 1 we present the basics of ultrametric spaces and valued fields. In Section 1.2 we quote several theorems on valued fields without proof (but with references) as we feel that otherwise it would lead us too far away from our main track. For that matter, the results are hardly needed, but may serve to offer the reader an impression.

Starting from Chapter 2 our policy changes: with obvious exceptions we present full proofs, as they really belong to the subject of the book. According to our experiences, in such cases not only the mere statements, but also their proofs, are needed in order to obtain understanding and intuition. We have also applied this philosophy to those parts in which the classical and the non-Archimedean theories seem to be similar. Not only does it facilitate a better grasp of the subtle differences but also serves those readers who may not be familiar with classical Functional Analysis theory (e.g. students and workers in other fields). In general, we hope that any reader, including the expert, will appreciate having the basic theory together with the proofs collected into a single volume.

In compiling the scattered material in the literature we were often able to simplify and tidy up the original proofs of the inventors; we hope that it will add to the value of the book. The same process revealed several natural questions that have not been touched upon before. We have tried to the best of our ability to fill in those gaps. Consequently this book contains quite a few new results.

To illustrate the theory we have included examples, mainly playing around spaces of continuous, analytic, and differentiable (C^n, C^∞) K-valued functions. The reader will notice that these themes return in every chapter, connecting them with the newly developed subject. For the reader's convenience we include a guide to the examples at the end of the book. Formally, the examples are independent of the main theory, so the reader can choose to pick his or her favourites; only occasionally will we use some space to provide a counterexample to a conjecture. We would like to point out that our examples do not cover all the present-day knowledge of these spaces. For this and related theories one should consult the following references.

Spaces of continuous functions were studied by e.g. Aguayo, De Grande-De Kimpe, Katsaras, Martínez-Maurica, Navarro and Perez-Garcia ([3], [5], [86], [87], [113], [114], [115], [120], [122], [123], [124], [149], [176]). Background on general theory of analytic functions was given by Escassut, [53], Robert, [190] and Robba and Christol, [188], [189]. The last reference also contains some locally convex examples and some applications to p-adic differential equations, as do [35] and [36]. More on locally convex theory of analytic functions can be found in [68], [78] and [82]. For $C^n(C^\infty)$-functions, see [195] for general theory, and the works of De Grande-De Kimpe, Khrennikov, Navarro, Schikhof and van Hamme ([70], [83], [165], [206], [207]) for locally convex aspects. Generalizations (several variables, more abstract settings) were treated by De Smedt ([223], [224], [225]) and by Bertram, Glöckner and Neeb ([28], [61]).

Throughout the book we have provided, for convenience, many cross-references. This way the reader can have easy access to related results that have been treated earlier, and a link to developments later on in the book.

The notes at the end of each chapter contain some history, references and comments. When mentioning results that are not covered by the book no completeness is pretended.

In the glossary of terms (Appendix A) we explain a few concepts, terminology, notations, etc., used freely in the book, but that may be not familiar to all readers.

In the guide to the examples (Appendix B) we list the most important examples treated in the book and indicate where their properties can be found.

Finally, we wish to thank the Ministerio de Educación y Ciencia of Spain (MTM2006-14786), for partially supporting the research concerning new results that are presented in this book.