# 1

# Ultrametrics and valuations

Ultrametric spaces will form the building blocks of the locally convex spaces to be treated in this book, whereas valued fields will act as their scalar fields.

We assume that the reader has a basic knowledge of ultrametrics and valuations; in this short chapter we will recall briefly some fundamentals that will be needed later on.

## 1.1 Ultrametric spaces

Let $X$ be a set. A *metric* on $X$ assigns to every ordered pair $(x, y) \in X \times X$ a nonnegative real number $d(x, y)$ such that:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality),

for all $x, y, z \in X$. The pair $(X, d)$ is called a *metric space*. We often write $X$ instead of $(X, d)$.

Let $(X, d)$ be a metric space. Let $a \in X$, $r > 0$. The set

$$B(a, r) := \{x \in X : d(x, a) \leq r\}$$

is called the *closed ball with radius $r$ about $a$*. (Indeed, $B(a, r)$ is closed in the induced topology.) Similarly,

$$B(a, r^-) := \{x \in X : d(x, a) < r\}$$

is called the *open ball with radius $r$ about $a$*.

In cases where we want to specify $X$ in the notation, we shall write $B_X(a, r)$ ($B_X(a, r^-)$) rather than $B(a, r)$ ($B(a, r^-)$). It is also customary to call the point $a$ the *centre of* the balls $B(a, r)$, $B(a, r^-)$. This will cause no harm as long as we keep in mind that a ball may have more than one centre (and more than one radius for that matter), see 1.1.2(b).

For a non-empty set $Y \subset X$ its *diameter* is

$$\operatorname{diam} Y := \sup\{d(x, y) : x, y \in Y\}$$

(possibly $\infty$). Analogously, the *distance between two non-empty sets* $Y, Z \subset X$ is

$$\operatorname{dist}(Y, Z) := \inf\{d(y, z) : y \in Y,\ z \in Z\}.$$

For $a \in X$ and $Y \subset X$, instead of dist $(\{a\}, Y)$ we write dist $(a, Y)$.

Occasionally we shall deal with *semi-metrics*, i.e., cases where condition (i) above is weakened to "$d(x, x) = 0$". Then the induced topology need not be Hausdorff. To avoid confusion we will use in this context sometimes expressions we left undefined (such as: $Y$ is $d$-dense in $X$) that yet will be clear to the reader.

The (semi-)metrics featuring in this book are of a special kind.

**Definition 1.1.1** Let $X = (X, d)$ be a metric space. $X$ is called an *ultrametric space*, and $d$ is called an *ultrametric*, if $d$ satisfies the *strong triangle inequality*:

(iii)′ $d(x, z) \le \max(d(x, y),\ \ d(y, z))$, for all $x, y, z \in X$.

Clearly (iii)′ is stronger than (iii).

At this point many examples of ultrametric spaces would fit here. However, most of them will be treated later on in their own right, so here we shall content ourselves with two examples.

**Examples 1.1.2**

(a) (A typical one: the $p$-adic metric) Let $p$ be a fixed prime. For $m, n \in \mathbb{Z}$, put $d(m, n) := 0$ if $m = n$ and, for $m \neq n$, $d(m, n) := p^{-r}$ if $r$ is the largest nonnegative integer such that $p^r$ divides $m - n$.

One checks easily that $d$ is an ultrametric on $\mathbb{Z}$. It may be worth mentioning that for $a \in \{0, 1, \ldots, p-1\}$ we have $B(a, 1^-) = B(a, p^{-1}) = \{n \in \mathbb{Z} : n \equiv a \pmod p\}$. Similarly, for $m \in \mathbb{N}$, $a \in \{0, 1, \ldots, p^m - 1\}$ we have $B(a, p^{-m}) = \{n \in \mathbb{Z} : n \equiv a \pmod{p^m}\}$. Thus, balls have a number-theoretic interpretation. Further, observe that the set $d(\mathbb{Z} \times \mathbb{Z})$ of values of $d$ equals $\{1, p^{-1}, p^{-2}, \ldots\} \cup \{0\}$. The induced topology is non-discrete. In fact, $\lim_n n! = 0$.

(b) (An extreme one: the trivial metric) Let $X$ be any set and for $x, y \in X$ put

$$d(x, y) := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Then $d$ is an ultrametric, called the *trivial metric*. It clearly induces the discrete topology. Notice that, for any $a \in X$ and for any $r \geq 1$, $X = B(a, r)$. So $X$ is a ball with infinitely many radii, and every point of $X$ serves as a centre.

Below we will state several basic facts on ultrametric spaces. Though simple to derive they are fundamental for all that follows.

**In the remaining part of this section $X = (X, d)$ is an ultrametric space.**

1. *Every point of a ball is a centre. Proof.* Consider the closed ball $B(a, r)$. Let $b \in B(a, r)$ and choose $x \in B(b, r)$. Then $d(x, a) \leq \max(d(x, b), d(b, a)) \leq r$, so $x \in B(a, r)$. By symmetry we have $B(a, r) = B(b, r)$. A similar proof works for open balls. ■

2. *Every ball is open and closed in the topological sense. Proof.* That a closed ball is open follows from 1. Now let $B$ be an open ball with radius $r$. The requirement "$x \sim y$ if and only if $d(x, y) < r$" defines an equivalence relation on $X$ whose classes are the open balls with radius $r$. Since $B$ is among them it follows that $X \setminus B$, being a union of open balls, is open, so $B$ is closed. ■

3. (Mercury drop behaviour) *Two balls are either disjoint, or one is contained in the other. Proof.* Let $B_1$, $B_2$ be two closed balls with radius $r_1, r_2$ respectively and let $a \in B_1 \cap B_2$. If $r_1 \leq r_2$ we have by 1, $B_1 = B(a, r_1) \subset B(a, r_2) = B_2$. A similar proof works for two open balls and for the case of an open and a closed ball. ■

4. (Isosceles triangle principle) Let $x, y, z \in X$. *Then among the numbers $d(x, z)$, $d(x, y)$, $d(y, z)$ the largest and second largest are equal. In other words, if $d(x, y) \neq d(y, z)$ then*

$$d(x, z) = \max(d(x, y), d(y, z)).$$

*Proof.* Suppose $d(x, y) < d(y, z)$. Then

$$d(x, z) \leq \max(d(x, y), d(y, z)) = d(y, z)$$

but also, since $d(y, z)$ is not $\leq d(x, y)$,

$$d(y, z) \leq \max(d(x, y), d(x, z)) = d(x, z),$$

and we are done. ■

5. (Equal distances between points of disjoint balls) *Let $B$ be a ball, $a \in X$, $a \notin B$. Then $d(a, x) = d(a, y)$ for each $x, y \in B$. More generally, let $B_1, B_2$ be disjoint balls. Then, for $x \in B_1$ and $y \in B_2$, the number $d(x, y)$ is constant.* We leave the proof to the reader.

6. (Some other immediate consequences) The topology of $X$ has a base of sets that are clopen, i.e., it is zero-dimensional and, by consequence, totally disconnected. For $a \in X$, $r > 0$, the "sphere" $\{x \in X : d(x, a) = r\}$ is clopen and, unless it is empty, *not* the boundary of $B(a, r)$. Another implication of the strong triangle inequality is that, to find the diameter of a non-empty set $Y \subset X$, it suffices to take an arbitrary $a \in Y$ and compute $\sup\{d(a, y) : y \in Y\}$. Also, a useful feature is the fact that if $B$ is a ball in $X$ and $Y$ is a subset of $X$ that meets $B$, then $B \cap Y$ is a ball in the metric space $Y$.

7. (No new values of the metric after completion) Let $x_1, x_2, \ldots$ be a sequence in $X$ converging to $x \in X$. Suppose $a \in X$, $a \neq x$. Then $d(x_n, a) = d(x, a)$ for large $n$. This follows easily from the isosceles triangle principle.

The metric completion of $X$ is again an ultrametric space, denoted $X^\wedge$, with the metric on $X^\wedge$ also denoted by $d$. By the above remark we have $d(X \times X) = d(X^\wedge \times X^\wedge)$.

8. Finally, a sequence $x_1, x_2, \ldots$ in $X$ is a Cauchy sequence if and only if $\lim_n d(x_{n+1}, x_n) = 0$. This follows from

$$d(x_m, x_n) \leq \max(d(x_m, x_{m-1}), \ldots, d(x_{n+1}, x_n)) \qquad (m > n).$$

For later use we include the following result.

**Theorem 1.1.3** *If $X$ is locally compact then there is a partition of $X$ consisting of compact balls.*

*Proof.* Choose $r_1, r_2, \ldots \in \mathbb{R}$ with $r_1 > r_2 > \cdots$, $\lim_n r_n = 0$. Let $\mathcal{B}$ be the collection of all closed balls in $X$ with radius in $\{r_1, r_2, \ldots\}$ that are compact. Then $\mathcal{B}$ is a covering of $X$ by compact balls. Every member of $\mathcal{B}$ is contained in a maximal member of $\mathcal{B}$. These maximal balls form the desired partition. ∎

The following concept will play a key role in this book.

**Definition 1.1.4** $X$ is called *spherically complete* if each nested sequence of balls $B_1 \supset B_2 \supset \cdots$ has a non-empty intersection.

We obtain equivalent definitions if in the above we require all $B_n$ to be closed balls, or all $B_n$ to be open balls.

It might be interesting to note that (ordinary) completeness of $X$ amounts to "each nested sequence $B_1 \supset B_2 \supset \cdots$ of balls for which $\lim_n \operatorname{diam} B_n = 0$ has a non-empty intersection". Thus, spherical completeness implies completeness. The converse is not true, which might seem surprising at first sight; for examples see 1.2.12 and 1.2.14.

Spherical completeness has to do with best approximations.

**Definition 1.1.5** Let $Y \subset X$, $a \in X$. We say that $a$ has a *best approximation in* $Y$ if $\operatorname{dist}(a, Y)$ is attained, i.e., $\min\{d(a, y) : y \in Y\}$ exists. Points in $Y$ having this minimal distance to $a$ are called *best approximations* (*of* $a$ *in* $Y$).

Note that in general there is more than one best approximation.

**Theorem 1.1.6** *Let $X \neq \varnothing$ be spherically complete. Let $Z$ be an ultrametric space containing $X$ as a subspace. Then each $z \in Z$ has a best approximation in $X$.*

*Proof.* There are $r_1 > r_2 > \cdots$ with $\lim_n r_n = \operatorname{dist}(z, X)$. Each one of the balls $B_n := \{x \in Z : d(x, z) \leq r_n\}$ has non-empty intersection with $X$, so $B_1 \cap X \supset B_2 \cap X \supset \cdots$ is a nested sequence of balls in $X$. Thus, there is an $a \in \bigcap_{n \in \mathbb{N}} (B_n \cap X)$. Clearly $d(z, a) = \operatorname{dist}(z, X)$. ■

## 1.2 Ultrametric fields

All fields appearing in this book are commutative. A *valuation on* a field $K$ is a map $|\,.\,| : K \longrightarrow [0, \infty)$ such that:

(i) $|\lambda| = 0$ if and only if $\lambda = 0$,
(ii) $|\lambda\, \mu| = |\lambda|\, |\mu|$ (multiplicativity),
(iii) $|\lambda + \mu| \leq |\lambda| + |\mu|$ (triangle inequality),

for all $\lambda, \mu \in K$. The pair $(K, |\,.\,|)$ is called a *valued field*. We often write $K$ instead of $(K, |\,.\,|)$. Note that, for all $\lambda, \mu \in K$, $\mu \neq 0$, we have $|-\lambda| = |\lambda|$, $\left|\lambda\, \mu^{-1}\right| = |\lambda| \,/\, |\mu|$ and $|1| = 1$ (here we adapt the bad habit to use the same symbol 1 for the unit element of $K$ and of $\mathbb{R}$; in fact, in (i) we did the same thing for the symbol 0).

Let $K$ be a valued field. Then the map $(\lambda, \mu) \mapsto |\lambda - \mu|$ is a metric on $K$ which induces a topology for which $K$ is a *topological field* (i.e., addition, subtraction, multiplication and division are continuous). The metric completion of $K$ is in a natural way a valued field which we denote by $K^\wedge$ (the unique continuous extension of the valuation is again usually denoted by $|\,.\,|$).

The (*closed*) *unit disk* (*or ball*) is the set $\{\lambda \in K : |\lambda| \le 1\} = B_K(0, 1)$, often abbreviated by $B_K$. Similarly, the *open unit disk* (*or ball*) is $B_K^- := \{\lambda \in K : |\lambda| < 1\} = B_K(0, 1^-)$. The set $B_K \setminus B_K^-$ is called the *unit sphere*.

Clearly $\mathbb{R}, \mathbb{C}$, with the absolute value function, henceforth denoted $|\,.\,|_\infty$, are examples of complete valued fields. $(\mathbb{Q}, |\,.\,|_\infty)$ is a non-complete valued field with completion $(\mathbb{R}, |\,.\,|_\infty)$.

**Definition 1.2.1** Two valuations on a field are called *equivalent* if they induce the same topology.

For example, the map $z \mapsto \sqrt{|z|_\infty}$ is a valuation on $\mathbb{C}$ that is equivalent to $|\,.\,|_\infty$.

**Theorem 1.2.2** *Two valuations $|\,.\,|_1$, $|\,.\,|_2$ on a field $K$ are equivalent if and only if there is an $s > 0$ such that $|\,.\,|_2 = |\,.\,|_1^s$.*

*Proof.* One direction is obvious; we sketch the proof of the "only if". For a valued field, the open unit disk is precisely the set $\{\lambda \in K : \lim_n \lambda^n = 0\}$, so $|\,.\,|_1$ and $|\,.\,|_2$ have the same open unit disk. By looking at $K \setminus B_K$ we see that $|\,.\,|_1$ and $|\,.\,|_2$ also have the same closed unit disk. Then also their unit spheres are the same. From mutiplicativity one infers the existence of a function $\phi$ for which $|\lambda|_2 = \phi(|\lambda|_1)$ for all $\lambda \in K$. From here it is an easy exercise to show that $\phi$ is a power function. ∎

In this book we focus on valued fields that are ultrametric.

**Definition 1.2.3** Let $K = (K, |\,.\,|)$ be a valued field. The valuation $|\,.\,|$ is called *non-Archimedean*, and $K$ is called a *non-Archimedean valued field* if $|\,.\,|$ satisfies the *strong triangle inequality*:

(iii)′ $|\lambda + \mu| \le \max(|\lambda|, |\mu|)$, for all $\lambda, \mu \in K$.

It is not our purpose to present here a complete theory of (non-Archimedean) valued fields. Instead we shall recall the most important facts and examples together to form either a refresher or a crash course, depending on the reader's point of view. One of the facts that we will use frequently is the identity $(1 - \lambda)^{-1} = \sum_{n=0}^{\infty} \lambda^n$ for $\lambda \in K$, $|\lambda| < 1$, where $K$ is a non-Archimedean valued field. We leave the proof of this familiar formula to the reader.

Let $(K, |\,.\,|)$ be a non-Archimedean valued field. Then, for each $n \in \mathbb{N}$,

$$|n.1| = |1 + 1 + \cdots + 1| \le \max(|1|, |1|, \ldots, |1|) = 1,$$

showing that the multiples of the unit element of $K$ form a bounded set. This is in contrast with the "axiom of Archimedes" stating that $\mathbb{N}$ is unbounded in $\mathbb{R}$ and, at the same time, it explains the term "non-Archimedean".

The isosceles triangle principle in $K$, "if $|\lambda| \neq |\mu|$ then $|\lambda + \mu| = \max(|\lambda|, |\mu|)$", shows that if $\lambda_1, \lambda_2, \ldots$ is a sequence in $K$ converging to $\lambda \neq 0$ then $|\lambda_n| = |\lambda|$ for large $n$. So no new values are added after completion.

We set $|K| := \{|\lambda| : \lambda \in K\}$ and $K^\times := K \setminus \{0\}$, the *multiplicative group of* $K$. Also, $\left|K^\times\right| := \{|\lambda| : \lambda \in K^\times\}$ is a multiplicative group of positive real numbers, the *value group of* $K$. There are two possibilities:

(i) 1 is not an accumulation point of $\left|K^\times\right|$. Then $\left|K^\times\right|$ is a discrete subset of $(0, \infty)$ and the valuation is called *discrete*.

It may happen that $\left|K^\times\right| = \{1\}$. Then we are dealing with the *trivial valuation* given by

$$|\lambda| := \begin{cases} 0 & \text{if } \lambda = 0 \\ 1 & \text{if } \lambda \in K^\times, \end{cases}$$

whose associated metric is the trivial one (see 1.1.2(b)) and whose induced topology is discrete.

If the valuation is not trivial but discrete then $\max\{|\lambda| : \lambda \in B_K^-\}$ exists, so there is a $\rho \in B_K^-$ such that $|K| \cap (|\rho|, 1) = \emptyset$. It is not difficult to see that $\left|K^\times\right| = \{|\rho|^n : n \in \mathbb{Z}\}$. We will call such a $\rho$ a *uniformizing element*.

(ii) 1 is an accumulation point of $\left|K^\times\right|$. Then $\left|K^\times\right|$ is a dense subset of $(0, \infty)$ and the valuation is called *dense*.

A second natural object that can be assigned to every non-Archimedean valued field is defined as follows. The unit disk $B_K$ of $K$ is not only multiplicatively, but, due to the strong triangle inequality, also additively closed. Thus, $B_K$ is a commutative ring with identity. The open unit disk $B_K^-$ is easily seen to be an ideal in $B_K$ and, since each element of $B_K \setminus B_K^-$ is invertible, even maximal. Thus, $B_K / B_K^-$ is a field, called the *residue class field of* $K$, customarily denoted by $k$. The canonical map $B_K \longrightarrow k$ is mostly written $\lambda \mapsto \overline{\lambda}$.

Now let $L$ be a *valued field extension of* $K$ (i.e., a non-Archimedean valued field containing $K$ as a valued subfield) with residue class fields $l$ and $k$ respectively. Consider the diagram

$$\begin{array}{ccc} B_K & \longrightarrow & B_L \\ \downarrow & & \downarrow \\ k & & l \end{array}$$

where the vertical arrows represent the canonical maps and the horizontal one is the inclusion. It is easily seen that there exists precisely one map $\phi : k \longrightarrow l$

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making the diagram

$$\begin{array}{ccc} B_K & \longrightarrow & B_L \\ \downarrow & & \downarrow \\ k & \xrightarrow[\phi]{} & l \end{array}$$

commute. Clearly $\phi$ is an injective ring homomorphism with $\phi(1) = 1$. We call this $\phi$ the *natural map* of $k$ into $l$.

Thus, we have that the *value group of $K$ is a subgroup of the value group of $L$ and that the residue class field of $K$ is naturally embedded into the residue class field of $L$.*

The next result is well known.

**Theorem 1.2.4** ([193], 1.5) *Let $K$ be a non-Archimedean valued field that is algebraically closed. Then its residue class field is again algebraically closed, its value group is divisible, and its valuation is either trivial or dense.*

**Example 1.2.5 The $p$-adic valuation on $\mathbb{Q}$**
Let $p$ be a prime number. Define a real-valued function $|\,.\,|_p$ on $\mathbb{Z}$ by $|0|_p := 0$ and, for $n \neq 0$,

$$|n|_p := p^{-r(n)},$$

where $r(n)$ is the largest nonnegative integer such that $p^{r(n)}$ divides $n$ (so, in fact $|n|_p = d(n, 0)$ of 1.1.2(a)). For a rational number $\frac{n}{m}$ ($m \neq 0$), set

$$\left|\frac{n}{m}\right|_p := \frac{|n|_p}{|m|_p}.$$

Direct verification shows that in this way a non-Archimedean valuation $|\,.\,|_p$ is defined on the field $\mathbb{Q}$ of rational numbers. $|\,.\,|_p$ is called the *$p$-adic valuation on $\mathbb{Q}$*.

We will show that – essentially – there are no other non-Archimedean valuations on $\mathbb{Q}$.

**Theorem 1.2.6** *Let $|\,.\,|$ be a non-Archimedean valuation on $\mathbb{Q}$. Then either $|\,.\,|$ is trivial or $|\,.\,|$ is equivalent to some $p$-adic valuation.*

*Proof.* Suppose $|\,.\,|$ is not trivial. Then $|n| \neq 1$ for some $n \in \mathbb{N}$. By the strong triangle inequality $|n| \leq 1$. Hence $\{m \in \mathbb{N} : |m| < 1\}$ is non-empty, let $p$ be its smallest element. Then $p \neq 1$. If $p = m\,n$ for some $m, n \in \mathbb{N}$, $m < p$, $n < p$ then $|m| = |n| = 1$, so $|p| = |m|\;|n| = 1$, an impossibility. Thus, $p$ is a prime number. If $n \in \mathbb{N}$ is not divisible by $p$ there exist $a \in \{0, 1, 2, \ldots\}$ and

$r \in \{1, \ldots, p-1\}$ such that $n = a\,p + r$. Then $|r| = 1$ and, by the isosceles triangle principle, also $|n| = 1$. Hence, for each $n \in \mathbb{N}$ we have $|n| = |p|^{r(n)}$, with $r(n)$ as above. It follows easily that, for all $\lambda \in \mathbb{Q}$, $|\lambda| = |\lambda|_p^s$, where $s := -\log\,|p|\,(\log p)^{-1}$. ■

$\mathbb{Q}$ is not complete with respect to the metric induced by the $p$-adic valuation. (The shortest way to see it is by means of a Baire Category argument, using the countability of $\mathbb{Q}$ and the fact that $(\mathbb{Q}, |\,.\,|_p)$ has no isolated points.)

**Definition 1.2.7** The completion of $(\mathbb{Q}, |\,.\,|_p)$ is called $\mathbb{Q}_p$, the *field of the p-adic numbers*. The extended valuation on $\mathbb{Q}_p$ is also denoted $|\,.\,|_p$.

Thus, in some natural sense, the fields $\mathbb{Q}_2, \mathbb{Q}_3, \ldots$ present themselves as alternatives to $\mathbb{R}$: completions of $\mathbb{Q}$, but with respect to the various $p$-adic valuations, which are obviously inequivalent.

The value group of $\mathbb{Q}_p$ is $\{p^n : n \in \mathbb{Z}\}$, so $|\,.\,|_p$ is discrete with uniformizing element $p \in \mathbb{Q}_p$. The unit disk is frequently denoted $\mathbb{Z}_p$ instead of $B_{\mathbb{Q}_p}$, and it is called the ring of *p-adic integers*; it is the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$. The residue class field of $\mathbb{Q}_p$ is the field of $p$ elements.

The fact that $(1-p)^{-1} = \sum_{n=0}^{\infty} p^n$ shows that there is no linear ordering in $\mathbb{Q}_p$ that behaves decently with respect to algebraic structure and topology.

$\mathbb{Q}_p$ is not algebraically closed. An easy way to see this is to consider the equation $x^2 = p$. Any solution $x$ would have the $p$-adic valuation $p^{-1/2}$, but this number is not in the value group. So $x^2 = p$ has no solutions in $\mathbb{Q}_p$. By considering the equation $x^n = p$ for any $n > 2$ we infer that the algebraic closure $\mathbb{Q}_p^a$ must be of infinite degree over $\mathbb{Q}_p$.

$\mathbb{Q}_p$ is locally compact, i.e., each point has a compact neighbourhood. In fact each ball in $\mathbb{Q}_p$, in particular $\mathbb{Z}_p$, is compact. This is a special case of the following.

**Theorem 1.2.8** ([195], 12.2) *Let $K$ be a non-trivially non-Archimedean valued complete field. Then $K$ is locally compact if and only if the valuation is discrete and the residue class field is finite.*

The following lemma will be needed in 2.5.22 and 2.5.34.

**Lemma 1.2.9** *Let $K$ be a non-trivially non-Archimedean valued field. Then $K$ contains a closed subfield $K_1$ that has a non-trivial discrete valuation.*

*Proof.* Let $K_0$ be the prime field of $K$. If it is non-trivially valued then $K_0 = \mathbb{Q}$ and the valuation on it is equivalent to a $p$-adic one (1.2.6), so is discrete and we can take $K_1 := \overline{K}_0$.

If $K_0$ is trivially valued, choose $a \in K$, $0 < |a| < 1$. By using the isosceles triangle principle we see that, for $\lambda_0, \lambda_1, \dots, \lambda_n \in K_0$,

$$\left|\lambda_0 + \lambda_1 a + \cdots + \lambda_n a^n\right| = \max_{0 \le i \le n} \left|\lambda_i a^i\right|.$$

So the elements $\frac{f}{g}$, where $f, g$ are polynomials in $a$ with coefficients in $K_0$, $g \neq 0$ form a valued subfield $K_0(a)$ of $K$ with discrete value group $\{|a|^n : n \in \mathbb{Z}\}$, and we can take $K_1 := \overline{K_0(a)}$. ■

Before giving the next example we first quote an extension theorem.

**Theorem 1.2.10** ([195], 14.1, 14.2) *Let $(K, |\,.\,|)$ be a non-Archimedean valued field. Let $L$ be a field extension of $K$, i.e., $L$ is a field containing $K$ as a subfield. Then $|\,.\,|$ can be extended to a non-Archimedean valuation on $L$. If, moreover, $K$ is complete and $L$ is of finite degree over $K$ (i.e., $L$ is finite-dimensional as a $K$-vector space) then this extension is unique.*

**Example 1.2.11 The field $\mathbb{C}_p$ of the $p$-adic complex numbers**
Each point of the algebraic closure $\mathbb{Q}_p^a$ of $\mathbb{Q}_p$ lies in a field extension of $\mathbb{Q}_p$ of finite degree over it, so by the previous theorem there is precisely one non-Archimedean valuation on $\mathbb{Q}_p^a$ that extends $|\,.\,|_p$. Unfortunately, $\mathbb{Q}_p^a$ is no longer complete ([195], 16.6), so we define $\mathbb{C}_p := (\mathbb{Q}_p^a)^\wedge$. Fortunately, the completion of an algebraically closed field is again algebraically closed ([195], 17.1). So $\mathbb{C}_p$ is in some sense the smallest complete algebraically closed extension of $\mathbb{Q}_p$, hence deserves the name *field of the $p$-adic complex numbers*. We will denote the extension of the $p$-adic valuation to $\mathbb{C}_p$ again by $|\,.\,|_p$.

We now quote some facts on $\mathbb{C}_p$.

**Theorem 1.2.12** ([195], 17.2, 20.6) *The value group of $\mathbb{C}_p$ is $\{p^r : r \in \mathbb{Q}\}$, its residue class field is the algebraic closure of the field of $p$ elements. So the valuation is dense and the residue class field is infinite. $\mathbb{C}_p$ is not locally compact, but still separable, complete but not spherically complete.*

That the field of the $p$-adic numbers is spherically complete follows from the next theorem.

**Theorem 1.2.13** *If $K$ is a discretely valued complete non-Archimedean field then $K$ is spherically complete.*

*Proof.* Let $B_1 \supset B_2 \supset \cdots$ be a nested sequence of balls in $K$, let $d_n := \operatorname{diam} B_n$. Then $d_1 \ge d_2 \ge \cdots$ and $d_n \in \left|K^\times\right|$ for each $n$. By discreteness either the sequence $B_1, B_2, \dots$ becomes stationary or $\lim_n d_n = 0$. For both cases (for the last one use completeness) we find $\bigcap_{n \in \mathbb{N}} B_n \neq \varnothing$. ■