Topology of Stratified Spaces
MSRI Publications
Volume **58**, 2011

An introduction to L^2 cohomology

XIANZHE DAI

ABSTRACT. After a quick introduction to L^2 cohomology, we discuss recent joint work with Jeff Cheeger where we study, from a mostly topological standpoint, the L^2-signature of certain spaces with nonisolated conical singularities. The contribution from the singularities is identified with a topological invariant of the link fibration of the singularities, involving the spectral sequence of the link fibration.

This paper consists of two parts. In the first, we give an introduction to L^2 cohomology. This is partly based on [8]. We focus on the analytic aspect of L^2 cohomology theory. For the topological story, we refer to [1; 22; 31] and of course the original papers [16; 17]. For the history and comprehensive literature, see [29]. The second part is based on our joint work with Jeff Cheeger [11], which gives the contribution to the L^2 signature from nonisolated conical singularity.

It is a pleasure to thank Eugenie Hunsicker for numerous comments and suggestions.

1. L^2 cohomology: what and why

What is L^2 cohomology? The de Rham theorem provides one of the most useful connections between the topological and differential structure of a manifold. The differential structure enters the de Rham complex, which is the cochain complex of smooth exterior differential forms on a manifold M, with the exterior derivative as the differential:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \to \cdots$$

Partially supported by NSF and CNSF..

The de Rham Theorem says that the de Rham cohomology, the cohomology of the de Rham complex, $H^k_{\mathrm{dR}}(M) \stackrel{\mathrm{def}}{=} \ker d_k/\mathrm{Im}\ d_{k-1}$, is isomorphic to the singular cohomology:

$$H^k_{\mathrm{dR}}(M) \cong H^k(M;\mathbf{R}).$$

The situation can be further rigidified by introducing geometry into the picture. Let g be a Riemannian metric on M. Then g induces an L^2-metric on $\Omega^k(M)$. As usual, let δ denote the formal adjoint of d. In terms of a choice of local orientation for M, we have $\delta = \pm * d *$, where $*$ is the Hodge star operator. Define the Hodge Laplacian to be

$$\Delta = d\delta + \delta d.$$

A differential form ω is harmonic if $\Delta\omega = 0$.

The great theorem of Hodge then states that, for a closed Riemannian manifold M, every de Rham cohomology class is represented by a unique harmonic form. This theorem provides a direct bridge between topology and analysis of manifolds through geometry, and has found many remarkable applications.

Naturally, then, one would like to extend the theory to noncompact manifolds and manifolds with singularity. The de Rham cohomology is still defined (one would restrict to the smooth open submanifold of a manifold with singularity). However, it does not capture the information at infinity or at the singularity.

One way of remedying this is to restrict to a subcomplex of the usual de Rham complex, namely that of the square integrable differential forms — this leads us to L^2 cohomology.

More precisely, let (Y, g) denote an open (possibly incomplete) Riemannian manifold, let $\Omega^i = \Omega^i(Y)$ be the space of C^∞ i-forms on Y and $L^2 = L^2(Y)$ the L^2 completion of Ω^i with respect to the L^2-metric. Define d to be the exterior differential with the domain

$$\mathrm{dom}\, d = \{\alpha \in \Omega^i(Y) \cap L^2(Y);\ d\alpha \in L^2(Y)\}.$$

Put

$$\Omega^i_{(2)}(Y) = \Omega^i(Y) \cap L^2(Y).$$

Then one has the cochain complex

$$0 \to \Omega^0_{(2)}(Y) \xrightarrow{d} \Omega^1_{(2)}(Y) \xrightarrow{d} \Omega^2_{(2)}(Y) \xrightarrow{d} \Omega^3_{(2)}(Y) \to \cdots.$$

The L^2-cohomology of Y is defined to be the cohomology of this cochain complex:

$$H^i_{(2)}(Y) = \ker d_i/\mathrm{Im}\, d_{i-1}\,.$$

Thus defined, the L^2 cohomology is in general no longer a topological invariant. However, the L^2 cohomology depends only on the quasi-isometry class of the metric.

EXAMPLES. • The real line: For the real line $\mathbb{R}$ with the standard metric,

$$H^i_{(2)}(\mathbb{R}) = 0 \quad \text{if } i = 0,$$
$$H^i_{(2)}(\mathbb{R}) \text{ is infinite-dimensional} \quad \text{if } i = 1.$$

For the first part, this is because constant functions can never be L^2, unless they are zero. For the second part, a 1-form $\phi(x)\,dx$, with $\phi(x)$ having compact support, is obviously closed and L^2, but can never be the exterior derivative of an L^2 function, unless the total integral of ϕ is zero.

• Finite cone: Let $C(N) = C_{[0,1]}(N) = (0,\ 1) \times N$, where N is a closed manifold of dimension n, with the conical metric $g = dr^2 + r^2 g_N$. Then a result of Cheeger [8] gives

$$H^i_{(2)}(C(N)) = \begin{cases} H^i(N) & \text{if } i < (n+1)/2, \\ 0 & \text{if } i \geq (n+1)/2. \end{cases}$$

Intuitively this can be explained by the fact that some of the differential forms that define classes for the cylinder $N \times (0,\ 1)$ cannot be L^2 on the cone if their degrees are too big. More specifically, let ω be an i-form on N and extend it trivially to $C(N)$, so ω is constant along the radial direction. Then

$$\int_{C(N)} |\omega|^2_g d\ \mathrm{vol}_g = \int_0^1 \int_N |\omega|_{g_N} r^{n-2i} dx\ dr.$$

Thus, the integral is infinite if $i \geq (n+1)/2$.

As we mentioned, the L^2 cohomology is in general no longer a topological invariant. Now clearly, there is a natural map

$$H^i_{(2)}(Y) \longrightarrow H^i(Y, \mathbf{R})$$

via the usual de Rham cohomology. However, this map is generally neither injective nor surjective. On the other hand, in the case when (Y, g) is a compact Riemannian manifold with corner (for a precise definition see the article by Gilles Carron in this volume), the map above is an isomorphism because the L^2 condition is automatically satisfied for any smooth forms.

Also, another natural map is from the compact supported cohomology to the L^2 cohomology:

$$H^i_c(Y) \longrightarrow H^i_{(2)}(Y).$$

As above, this map is also neither injective nor surjective in general.

Instead, the L^2 cohomology of singular spaces is intimately related to the intersection cohomology of Goresky–MacPherson ([16; 17]; see also Greg Friedman's article in this volume for the intersection cohomology). This connection was pointed out by Dennis Sullivan, who observed that Cheeger's local computation of L^2 cohomology for isolated conical singularity agrees with that of

Goresky–MacPherson for the middle intersection homology. In [8], Cheeger established the isomorphism of the two cohomology theories for admissible pseudomanifolds. One of the fundamental questions has been the topological interpretation of the L^2 cohomology in terms of the intersection cohomology of Goresky–MacPherson.

Reduced L^2 cohomology and L^2 harmonic forms. In analysis, one usually works with complete spaces. That means, in our case, the full L^2 space instead of just smooth forms which are L^2. Now the coboundary operator d has well defined strong closure $\bar{d}$ in L^2: $\alpha \in \operatorname{dom} \bar{d}$ and $\bar{d}\alpha = \eta$ if there is a sequence $\alpha_j \in \operatorname{dom} d$ such that $\alpha_j \to \alpha$ and $d\alpha_j \to \eta$ in L^2. (The strong closure is to make $\bar{d}$ a closed operator. There are other notions of closures and extensions, as in [15] for instance.) Similarly, δ has the strong closure $\bar{\delta}$.

One can also define the L^2-cohomology using the strong closure $\bar{d}$. Thus, define

$$H^i_{(2),\#}(Y) = \ker \bar{d}_i / \operatorname{Im} \bar{d}_{i-1} .$$

Then the natural map

$$\iota_{(2)} : \; H^i_{(2)}(Y) \longrightarrow H^i_{(2),\#}(Y)$$

turns out to be always an isomorphism [8].

This is good, but does not produce any new information ... yet! The crucial observation is that, in general, the image of $\bar{d}$ need not be closed. This leads to the notion of reduced L^2-cohomology, which is defined by quotienting out by the closure instead:

$$\bar{H}^i_{(2)}(Y) = \ker \bar{d}_i / \overline{\operatorname{Im} \bar{d}_{i-1}} .$$

The reduced L^2-cohomology is generally not a cohomology theory but it is intimately related to Hodge theory, as we will see.

Now we define the space of L^2-harmonic i-forms $\mathcal{H}^i_{(2)}(Y)$ to be

$$\mathcal{H}^i_{(2)}(Y) = \{\theta \in \Omega^i \cap L^2; d\theta = \delta\theta = 0\}.$$

Some authors define the L^2-harmonic forms differently; compare [31]. The definitions coincide when the manifold is complete. The advantage of our definition is that, when Y is oriented, the Hodge star operator induces

$$* \; : \mathcal{H}^i_{(2)}(Y) \to \mathcal{H}^{n-i}_{(2)}(Y),$$

which is naturally the Poincaré duality isomorphism.

Now the big question is: Do we still have a Hodge theorem?

Kodaira decomposition, L^2 Stokes and Hodge theorems. To answer the question, let's look at the natural map, the Hodge map

$$\mathcal{H}^i_{(2)}(Y) \longrightarrow H^i_{(2)}(Y).$$

The question becomes: When is this map an isomorphism? Following Cheeger [8], when the Hodge map is an isomorphism, we will say that the strong Hodge theorem holds.

The most basic result in this direction is the Kodaira decomposition [23] (see also [14]),

$$L^2 = \mathcal{H}^i_{(2)} \oplus \overline{d\Lambda_0^{i-1}} \oplus \overline{\delta\Lambda_0^{i+1}},$$

an orthogonal decomposition which leaves invariant the subspaces of smooth forms. Here subscript 0 denotes having compact support. This result is essentially the elliptic regularity.

It follows from the Kodaira decomposition that

$$\ker \bar{d}_i = \mathcal{H}^i_{(2)} \oplus \overline{d\Lambda_0^{i-1}}.$$

Therefore the question reduces to what the space $\operatorname{Im} \bar{d}_{i-1}$ is in the decomposition. We divide the discussion into two parts:

SURJECTIVITY. If $\operatorname{Im} \bar{d}$ is closed, then $\operatorname{Im} \bar{d} \supset \overline{d\Lambda_0^{i-1}}$. Hence, the Hodge map is surjective in this case.

In particular, this holds if the L^2-cohomology is finite-dimensional.

INJECTIVITY. The issue of injectivity of the Hodge map has to do with the L^2 Stokes theorem. We say that Stokes' theorem holds for Y in the L^2 sense if

$$\langle \bar{d}\alpha,\ \beta\rangle = \langle \alpha,\ \bar{\delta}\beta\rangle$$

for all $\alpha \in \operatorname{dom} \bar{d}$, $\beta \in \operatorname{dom} \bar{\delta}$; or equivalently, if

$$\langle d\alpha,\ \beta\rangle = \langle \alpha,\ \delta\beta\rangle$$

$\alpha \in \operatorname{dom} d$, $\beta \in \operatorname{dom} \delta$.

If the L^2 Stokes theorem holds, one has

$$\mathcal{H}^i_{(2)}(Y) \perp \operatorname{Im} \bar{d}_{i-1},$$

so the Hodge map is injective in this case. Moreover,

$$H^i_{(2)}(Y) = \mathcal{H}^i_{(2)}(Y) \oplus \overline{\operatorname{Im} \bar{d}_{i-1}} / \operatorname{Im} \bar{d}_{i-1}.$$

Here, by the closed graph theorem, the last summand is either 0 or infinite-dimensional. Note also, since it follows that

$$\mathcal{H}^i_{(2)}(Y) \perp \overline{\operatorname{Im} \bar{d}_{i-1}},$$

that

$$\bar{H}^i_{(2)}(Y) \cong \mathcal{H}^i_{(2)}(Y).$$

That is, when the L^2 Stokes theorem holds, the reduced L^2 cohomology is simply the space of L^2 harmonic forms.

Summarizing, if the L^2-cohomology of Y has finite dimension and Stokes' theorem holds on Y in the L^2-sense, then the Hodge theorem holds in this case, and the L^2-cohomology of Y is isomorphic to the space of L^2-harmonic forms. Therefore, when Y is orientable, Poincaré duality holds as well. Consequently, the L^2 signature of Y is well defined in this case.

There are several now classical results regarding the L^2 Stokes theorem. Gaffney [15] showed that the L^2 Stokes theorem holds for complete Riemannian manifolds. On the other hand, for manifolds with conical singularity $M = M_0 \cup C(N)$, the general result of Cheeger [9] says that the L^2 Stokes theorem holds provided that L^2 Stokes holds for N and in addition the middle-dimensional (L^2) cohomology group of N vanishes if dim N is even. In particular, if N is a closed manifold of odd dimension, or $H^{\dim N/2}(N) = 0$ if dim N is even, the L^2 Stokes theorem holds for M.

REMARK. There are various extensions of L^2, for cohomology example, cohomology with coefficients or Dolbeault cohomology for complex manifolds.

2. L^2 signature of nonisolated conical singularities

Nonisolated conical singularities. We now consider manifolds with nonisolated conical singularity whose strata are themselves smooth manifolds. In other words, singularities are of the following type:

(i) The singular stratum consists of disjoint unions of smooth submanifolds.
(ii) The singularity structure along the normal directions is conical.

More precisely, a neighborhood of a singular stratum of positive dimension can be described as follows. Let

$$Z^n \to M^m \xrightarrow{\pi} B^l \tag{2-1}$$

be a fibration of closed oriented smooth manifolds. Denote by $C_\pi M$ the mapping cylinder of the map $\pi : M \to B$. This is obtained from the given fibration by attaching a cone to each of the fibers. Indeed, we have

$$C_{[0,1]}(Z) \to C_\pi M \to B\,.$$

The space $C_\pi M$ also comes with a natural quasi-isometry class of metrics. A metric can be obtained by choosing a submersion metric on M:

$$g_M = \pi^* g_B + g_Z.$$

Then, on the nonsingular part of $C_\pi M$, we take the metric

$$g_1 = dr^2 + \pi^* g_B + r^2 g_Z. \tag{2-2}$$

The general class of spaces with nonisolated conical singularities as above can be described as follows. A space X in the class will be of the form

$$X = X_0 \cup X_1 \cup \cdots \cup X_k,$$

where X_0 is a compact smooth manifold with boundary, and each X_i (for $i = 1, \ldots, k$) is the associated mapping cylinder $C_{\pi_i} M_i$ for some fibration (M_i, π_i), as above.

More generally, one can consider the iterated construction where we allow manifolds in our initial fibration to have singularities of the type considered above. However, we will restrict ourselves to the simplest situation where the initial fibrations are all modeled on smooth manifolds.

Remark. An n-dimensional stratified pseudomanifold X is a topological space together with a filtration by closed subspaces

$$X = X_n = X_{n-1} \supset X_{n-2} \supset \cdots \supset X_1 \supset X_0$$

such that for each point $p \in X_i - X_{i-1}$ there is a distinguished neighborhood U in X which is filtered homeomorphic to $C(L) \times B^i$ for a compact stratified pseudomanifold L of dimension $n-i-1$. The *i-dimensional stratum* $X_i - X_{i-1}$ is an i-dimensional manifold. A *conical metric* on X is a Riemannian metric on the regular set of X such that on each distinguished neighborhood it is quasi-isometric to a metric of the type (2-2) with $B = B^i$, $Z = L$ and g_B the standard metric on B^i, g_Z a conical metric on L. Such conical metrics always exist on a stratified pseudomanifold.

L^2 signature of generalized Thom spaces. A generalized Thom space T is obtained by coning off the boundary of the space $C_\pi M$.

Namely,

$$T = C_\pi M \cup_M C(M)$$

is a compact stratified pseudomanifold with two singular strata: B and a single point (unless B is a sphere).

Example. Let $\xi \xrightarrow{\pi} B$ be a vector bundle of rank k. We have the associated sphere bundle

$$S^{k-1} \to S(\xi) \xrightarrow{\pi} B.$$

The generalized Thom space constructed out of this fibration coincides with the usual Thom space equipped with a natural metric.

Now consider the generalized Thom space constructed from an oriented fibration (2-1) of closed manifolds, i.e., both the base B and fiber Z are closed oriented manifolds and so is the total space M. Then T will be a compact oriented stratified pseudomanifold with two singular strata. Since we are interested in the L^2 signature, we assume that the dimension of M is odd (so $\dim T$ is even). In addition, we assume the Witt conditions; namely, either the dimension of the fibers is odd or its middle-dimensional cohomology vanishes. Under the Witt conditions, the strong Hodge theorem holds for T. Hence the L^2 signature of T is well defined.

QUESTION. What is the L^2 signature of T?

Let's go back to the case of the usual Thom space.

EXAMPLE (continued). In this case,

$$\operatorname{sign}_{(2)}(T) = -\operatorname{sign}(D(\xi)),$$

the signature of the disk bundle $D(\xi)$ (as a manifold with boundary).

Let Φ denote the Thom class and χ the Euler class. Then the Thom isomorphism gives the commutative diagram

$$\begin{array}{ccccc} H^{*+k}(D(\xi), S(\xi)) & \otimes & H^{*+k}(D(\xi), S(\xi)) & \longrightarrow & \mathbb{R} \\ \uparrow {\scriptstyle \pi^*(\cdot)\cup\Phi} & & \uparrow {\scriptstyle \pi^*(\cdot)\cup\Phi} & & \\ H^*(B) & \otimes & H^*(B) & \longrightarrow & \mathbb{R} \\ & & & & \\ \phi & & \psi & \longrightarrow & [\phi\cup\psi\cup\chi][B]. \end{array}$$

Thus, $\operatorname{sign}_{(2)}(T)$ is the signature of this bilinear form on $H^*(B)$.

We now introduce the topological invariant which gives the L^2-signature for a generalized Thom space. In [13], in studying adiabatic limits of eta invariants, we introduced a global topological invariant associated with a fibration. (For adiabatic limits of eta invariants, see also [32; 5; 10; 3].) Let (E_r, d_r) be the E_r-term with differential, d_r, of the Leray spectral sequence of the fibration (2-1) in the construction of the generalized Thom space T. Define a pairing

$$\begin{aligned} E_r \otimes E_r &\longrightarrow \mathbb{R} \\ \phi \otimes \psi &\longmapsto \langle \phi \cdot d_r \psi, \xi_r \rangle, \end{aligned}$$

where ξ_r is a basis for E_r^m naturally constructed from the orientation. In case $m = 4k-1$, when restricted to $E_r^{(m-1)/2}$, this pairing becomes symmetric. We define τ_r to be the signature of this symmetric pairing and put

$$\tau = \sum_{r \geq 2} \tau_r \,.$$

When the fibration is a sphere bundle with the typical fiber a $(k-1)$-dimensional sphere, the spectral sequence satisfies $E_2 = \cdots = E_k$, $E_{k+1} = E_\infty$ with $d_2 = \cdots = d_{k-1} = 0$, $d_k(\psi) = \psi \cup \chi$. Hence τ coincides with the signature of the bilinear form from the Thom isomorphism theorem. The main result of [11] is this:

THEOREM 1 (CHEEGER–DAI). *Assume either the fiber Z is odd-dimensional or its middle-dimensional cohomology vanishes. Then the L^2-signature of the generalized Thom space T is equal to $-\tau$:*

$$\operatorname{sign}_{(2)}(T) = -\tau.$$

In spirit, our proof of the theorem follows the example of the sphere bundle of a vector bundle. Thus, we first establish an analog of Thom's isomorphism theorem in the context of generalized Thom spaces. In part, this consists of identifying the L^2-cohomology of T in terms of the spectral sequence of the original fibration; see [11] for complete details.

COROLLARY 2. *For a compact oriented space X with nonisolated conical singularity satisfying the Witt conditions, the L^2-signature is given by*

$$\operatorname{sign}_{(2)}(X) = \operatorname{sign}(X_0) + \sum_{i=1}^{k} \tau(X_i) \,.$$

The study of the L^2-cohomology of the type of spaces with conical singularities discussed here turns out to be related to work on the L^2-cohomology of noncompact hyperkähler manifolds which is motivated by Sen's conjecture; see, for example, [19] and [18]. Hyperkähler manifolds often arise as moduli spaces of (gravitational) instantons and monopoles, and so-called S-duality predicts the dimension of the L^2-cohomology of these moduli spaces (Sen's conjecture). Many of these spaces can be compactified to give a space with nonisolated conical singularities. In such cases, our results can be applied. We also refer the reader to the work [18] of Hausel, Hunsicker and Mazzeo, which studies the L^2-cohomology and L^2-harmonic forms of noncompact spaces with fibered geometric ends and their relation to the intersection cohomology of the compactification. Various applications related to Sen's conjecture are also considered there.

Cambridge University Press
978-0-521-19167-8 - Topology of Stratified Spaces
Edited by Greg Friedman, Eugénie Hunsicker, Anatoly Libgober and Laurentiu Maxim
Excerpt
More information

Combining the index theorem of [4] with our topological computation of the L^2-signature of T, we recover the following adiabatic limit formula of [13]; see also [32; 5; 9; 3].

COROLLARY 3. *Assume that the fiber Z is odd-dimensional. Then we have the following adiabatic limit formula for the eta invariant of the signature operator*:

$$\lim_{\varepsilon\to 0} \eta(A_{M,\varepsilon}) = \int_B \mathcal{L}\left(\frac{R^B}{2\pi}\right) \wedge \tilde{\eta} + \tau.$$

In the general case, that is, with no dimension restriction on the fiber, the L^2-signature for generalized Thom spaces is discussed in [21]. In particular, Theorem 1 is proved for the general case in [21]. However, one of the ingredients there is the adiabatic limit formula of [13], rather than the direct topological approach taken here. One of our original motivations was to give a simple topological proof of the adiabatic limit formula. In [20], the methods and techniques in [11] are used in the more general situation to derive a very interesting topological interpretation for the invariant τ_r. On the other hand, in [7], our result on the generalized Thom space, together with the result in [13], is used to derive the signature formula for manifolds with nonisolated conical singularity.

References

[1] A. Borel et al. *Intersection cohomology*. Notes on the seminar held at the University of Bern, Bern, 1983. Reprint of the 1984 edition. Birkhäuser, Boston, 2008.

[2] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and riemannian geometry, I–III. *Math. Proc. Cambridge Philos. Soc.* 77(1975):43–69, 78(1975):405–432, 79(1976):71–99.

[3] J.-M. Bismut and J. Cheeger. η-invariants and their adiabatic limits. *Jour. Amer. Math. Soc.* 2:33–70, 1989.

[4] J.-M. Bismut and J. Cheeger. Remarks on families index theorem for manifolds with boundary. *Differential geometry*, 59–83, Pitman Monogr. Surveys Pure Appl. Math., 52, Longman Sci. Tech., Harlow, 1991 eds. Blaine Lawson and Keti Tenenbaum.

[5] J.-M. Bismut and D. S. Freed. The analysis of elliptic families I, II. *Commun. Math. Phys.* 106:159–167, 107:103–163, 1986.

[6] R. Bott and L. Tu. *Differential forms in algebraic topology*. Graduate Texts in Mathematics, 82. Springer, New York-Berlin, 1982.

[7] J. Brüning. The signature operator on manifolds with a conical singular stratum. *Astérisque*, to appear.

[8] J. Cheeger. On the Hodge theory of Riemannian pseudomanifolds. *Geometry of the Laplace operator*, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, R.I. 36:91–146, 1980.