1 RF components

In this chapter basic components used in RF design are discussed. A detailed modeling and analysis of MOS transistors at high frequency may be found in [1], [2]. Although mainly developed for analog and high-speed circuits, the model is good enough for most RF applications operating at several GHz, especially for the nanometer CMOS processes used today. Thus, we will offer a more detailed study of inductors, capacitors, and LC resonators instead in this chapter. We will also present a brief discussion on the fundamental operation of distributed circuits and transmission lines, and follow up on that in Chapter 3. In Chapters 4 and 6, we will discuss some of the RF related aspects of transistors, such as more detailed noise analysis as well as substrate and gate resistance impact.

LC circuits are widely used in RF design, with applications ranging from tuned amplifiers, matching circuits, and LC oscillators. Inspired by superior noise and linearity compared to transistors, historically, radios have relied heavily on inductors and capacitors, with large portions of the RF blocks occupied by them. Although this dependence has been reduced in modern radios, mostly for cost concerns, still RF designers deal with integrated inductors and capacitors quite often.

We start the chapter with a brief introduction to electromagnetic fields, and take a closer look at capacitors and inductors from the electromagnetic field perspective. We will then discuss capacitors, inductors, and LC resonators from a circuit point of view. We conclude the chapter by presenting the principles and design tradeoffs of integrated inductors and capacitors.

The specific topics covered in this chapter are:

- capacitance and inductance electromagnetic and circuit definitions;
- Maxwell's equations;
- distributed elements and introduction to transmission lines;
- energy, power, and quality factor;
- lossless and low-loss resonance circuits;
- integrated capacitors and inductors.

For class teaching, we recommend focusing on Sections 1.7, 1.8, and 1.9, while Sections 1.1–1.6 may be assigned as reading, or only a brief summary presented if deemed necessary.

1.1 ELECTRIC FIELDS AND CAPACITANCE

Let us start with a brief overview of electric fields and electric potential. We shall define the concept of capacitance accordingly.

Published first in 1875 by Charles Coulomb, the French army officer, Coulomb's law states that the force between two point charges, separated in vacuum or free space by a distance, is proportional to each charge, and inversely proportional to the square of the distance between them (Figure 1.1). It bears a great similarity to Newton's gravitational law, discovered about a hundred years earlier. Writing the force ($\boldsymbol{F}_\mathrm{t}$) as a force per unit charge gives the electric field intensity, $\boldsymbol{E}$ measured in V/m (or volt per meter) as follows:

$$\boldsymbol{E} = \frac{\boldsymbol{F}_\mathrm{t}}{Q_\mathrm{t}} = \frac{Q}{4\pi\epsilon_0 r^2}\boldsymbol{a}_r,$$

where the ***bold*** notation indicates a vector in 3D space, $\epsilon_0 = \frac{1}{36\pi} \times 10^{-9}$ F/m (or farads per meter) is the permittivity in free space, and Q is the charge in C (or coulomb).[1] $\boldsymbol{a}_r$ is the unit vector pointing in the direction of the field, which is in the same direction as the vector $\boldsymbol{r}$ connecting the charge Q to the point of interest P in space (see Figure 1.1). Q_t is a test charge to which the force (or field) created by Q is applied.

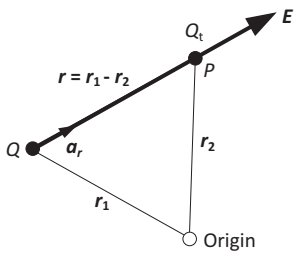

Figure 1.1 Coulomb's law

In many cases the electric field can be calculated more easily by applying Gauss's law instead. It states that the electric flux density,[2] $\boldsymbol{D} = \epsilon_0\boldsymbol{E}$ (measured in C/m^2) passing through any closed surface is equal to the total charge enclosed by that surface,[3] and mathematically expressed as:

$$\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = Q,$$

where $\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S}$ indicates the integral over a closed surface. The *dot* product ($\cdot$) indicates the product of the magnitude and the cosine of the smaller angle. The charge Q could be the sum of several charge points, that is: $Q = \sum Q_i$, a volume charge distribution: $Q = \int_V \rho_V \mathrm{d}V$, or a surface distribution, . . . The nature of the surface integral implies that only the normal component of $\boldsymbol{D}$ at the surface contributes to the charge, whereas the tangential component leads to $\boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S}$ equal to zero.

[1] Not to be confused with Q used as quality factor later in this chapter. [2] Only in free space.
[3] The expression itself is a result of Michael Faraday's experiment. Gauss's contribution was providing the mathematical tools to formulate it.

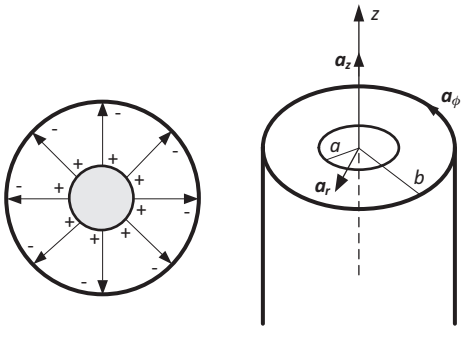

Figure 1.2 Electric flux in a coaxial cable

For example, consider a long coaxial cable with inner radius a and outer radius b, carrying a uniform charge distribution ρ_S on the outer surface of the inner conductor (and $-\frac{b}{a}\rho_S$ on the inner surface of the outer conductor) as shown in Figure 1.2. For convenience, let us use cylindrical coordinates [3].

The flux will have components in the $\boldsymbol{a}_r$ direction, normal to the surface. For an arbitrary length L in the z-axis direction, we can write

$$\int_{z=0}^{L}\int_{\phi=0}^{2\pi} D_r\,(r\mathrm{d}\phi\mathrm{d}z) = Q = \rho_S(2\pi a L).$$

Thus, inside the cable, that is for $a < r < b$:

$$\boldsymbol{D} = \frac{\rho_S a}{r}\boldsymbol{a}_r.$$

The electric field and flux density are both zero outside the cable as the net charge is equal to zero.

Based on the electric energy definition,[4] the potential difference between points A and B (V_{AB}) is defined as

$$V_{\mathrm{AB}} = \frac{W}{Q} = -\int_{\mathrm{B}}^{\mathrm{A}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{L},$$

where W is the energy in J (or joule), and the right side is the *line integral* of the electric field. The physical interpretation of potential is such that moving a charge Q along with the electric field from point A to B results in energy reduction (or the charge releases energy), and accordingly we expect point A to be at a *higher potential*. By definition of the line integral, the sum of *static* potentials in a closed path must be equal to zero, that is to say $\oint \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{L} = 0$, which is a general representation of *Kirchhoff's voltage law* or KVL. This is physically understood by noting that when the charge is moved around a closed path, the total energy received and released balance each other, thus no *net* work is done.

[4] We shall discuss the electric energy shortly.

We close this section by defining the capacitance. Suppose we have two oppositely charged (each with a charge of Q) conductors M1 and M2 within a given dielectric with permittivity[5] of $\epsilon = \epsilon_r \epsilon_0$ (Figure 1.3). Assuming a potential difference of V_0 between the conductors, we define capacitance C measured in Farad as

$$C = \frac{Q}{V_0}.$$

Alternatively, one can re-write C as

$$C = \epsilon \frac{\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S}}{-\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{L}},$$

which indicates that capacitance (if linear), is independent of the charge or potential, as $\boldsymbol{E}$ (or $\boldsymbol{D}$) depends linearly on Q according to Gauss's law.

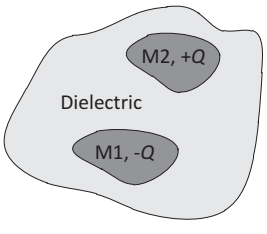

Figure 1.3 Capacitance definition

Physically, the capacitance indicates the capability of energy or equivalently electric flux storage in electrical systems, analogous to inductors that store magnetic flux.

Returning to our previous example of the coaxial cable, the potential between the inner and outer conductors is calculated by taking the line integral of $\boldsymbol{E} = \boldsymbol{D}/\epsilon$, where $\boldsymbol{D}$ was obtained previously. This yields

$$V_0 = -\frac{1}{\epsilon} \int_b^a \frac{a\rho_S}{r} \mathrm{d}r = \frac{a\rho_S \ln\frac{b}{a}}{\epsilon},$$

and thus the capacitance per unit length is equal to

$$C = \frac{2\pi\epsilon}{\ln\frac{b}{a}}.$$

Clearly, the capacitance is only a function of the coaxial cable radii and the dielectric.

1.2 MAGNETIC FIELDS AND INDUCTANCE

A steady magnetic field can be created in one of three ways: through a permanent magnet, a linear time-varying electric field, or simply due to a direct current. The permanent magnet has several applications in RF and microwave devices, such as passive gyrators used in a lossless circulator, which is a *passive*, but *non-reciprocal* circuit, [4], [5]. However, we will mostly focus

[5] $\epsilon_r = 1$ for free space.

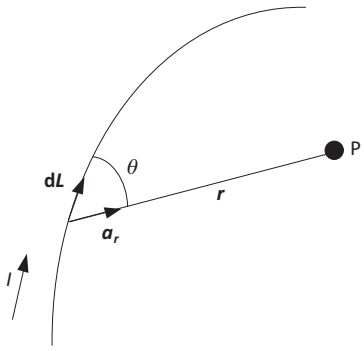

Figure 1.4 Biot–Savart law expression

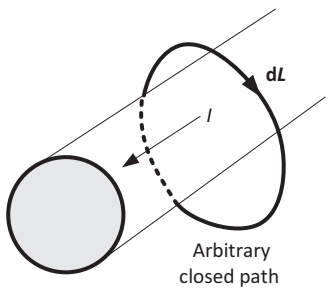

Figure 1.5 Ampère's law

on the latter two methods of creating the magnetic fields, and defer the gyrator and circulator discussion to [4].

In 1820, the law of Biot-Savart was proposed as follows, which associates magnetic field intensity $\boldsymbol{H}$ (expressed in A/m) at a given point P to a current of I flowing in a differential vector length d$\boldsymbol{L}$ of an ideal filament (Figure 1.4):

$$\mathrm{d}\boldsymbol{H} = \frac{I\,\mathrm{d}\boldsymbol{L} \times \boldsymbol{a}_r}{4\pi r^2}.$$

The *cross* product ($\times$) indicates the product of the magnitude and the sine of the smaller angle. The magnetic field will then be perpendicular to the plane containing the current filament and the vector $\boldsymbol{r}$, and whose direction is determined based on the right hand rule. The law states that the magnetic field intensity is directly proportional to the current (I), but inversely proportional to the square of the distance (r) between P and the differential length, and also proportional to the magnitude of the differential element times the sine of the angle θ shown in Figure 1.4.

A more familiar law describing the magnetic field was proposed by Ampère shortly after in 1823, widely known as Ampère's circuital law,[6] and is mathematically expressed as:

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{L} = I,$$

indicating that the line integral of the magnetic field ($\boldsymbol{H}$) about any *closed path* is exactly equal to the current enclosed by that path (Figure 1.5). This law proves to be more useful as it allows us

[6] Ampère's law may be derived from Biot-Savart's law.

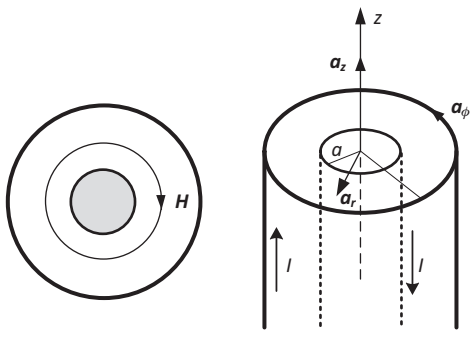

Figure 1.6 Magnetic field in a coaxial cable, top and side views

to calculate the magnetic field more easily as long as which components of the field are present is properly determined, and the symmetry is invoked appropriately. By comparison, Ampère's circuital law is more analogous to Gauss's law, whereas the law of Biot-Savart could be considered similar to Coulomb's law.

As an example, consider a long coaxial cable carrying a uniform current of I in the center conductor and $-I$ in the outer one, as shown in Figure 1.6. Clearly the field cannot have any component in the z direction, as it must be normal to the current direction. Moreover, the symmetry shows that $\boldsymbol{H}$ cannot be a function of ϕ or z, and thus could be expressed as a general form of $\boldsymbol{H} = H_r \boldsymbol{a}_\phi$. Inside the coaxial cable, that is $a < r < b$, applying the line integral then leads to

$$\boldsymbol{H} = \frac{I}{2\pi r}\boldsymbol{a}_\phi.$$

Moreover, similarly to the electric field, the magnetic field is zero outside the cable as the net current flow is zero, showing the concept of *shielding* provided by the coaxial cable. Note that, inside the cable, the magnetic field consists of closed lines circling around the current, as opposed to the electric field lines that start on a positive charge and end on a negative one.

In free space, magnetic flux density $\boldsymbol{B}$ (measured in Webber/m^2 or tesla), is defined as

$$\boldsymbol{B} = \mu_0 \boldsymbol{H},$$

where $\mu_0 = 4\pi \times 10^{-7}$ H/m (or Henry per meter) in free space and is the *permeability*. The magnetic flux, ϕ, is then the flux passing through a designated area S, measured in Webber, and is defined as

$$\phi = \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}.$$

Generally the magnetic flux is a linear function of the current (I), that is, $\phi = LI$, where the proportionality constant, L, is known as the *inductance*, and is measured in Henry. We can thus say:

$$L = \mu_0 \frac{\int_S \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{S}}{\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{L}},$$

and since $\boldsymbol{H}$ is a linear function of I, as established by Ampère's (or Biot-Savart's) law, the inductance is a function of the conductor geometry and the distribution of the current, but not

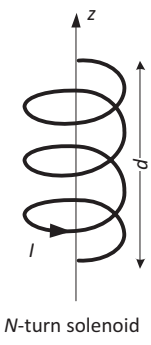

Figure 1.7 An N-turn solenoid

the current itself. As an example, calculating the total flux inside the coaxial cable of the previous example, one can show that the cable inductance per unit length is

$$L = \frac{\mu_0}{2\pi} \ln \frac{b}{a},$$

whereas the capacitance per unit length of the same coaxial cable was calculated before by applying Gauss's law, and is equal to

$$C = \frac{2\pi\epsilon}{\ln \frac{b}{a}}.$$

Clearly,

$$LC = \mu_0 \epsilon.$$

We conclude this section by defining the mutual inductance M_{12} between circuits 1 and 2 in terms of their *flux linkage*:

$$M_{12} = \frac{N_2 \phi_{12}}{I_1},$$

where ϕ_{12} signifies the flux produced by I_1 which links the path of the filamentary current I_2, and N_2 is the number of turns in circuit 2. The mutual inductance therefore depends on the magnetic interaction between the two currents.

As an example, consider an N-turn solenoid with finite length d, consisting of N closely wound filaments that carry a current I, as shown in Figure 1.7. We assume the solenoid is long relative to its diameter.

The magnetic field is in the $\boldsymbol{a}_z$ direction, as the current is in the $\boldsymbol{a}_\phi$ direction, and Ampère's law readily shows that within the solenoid,

$$\boldsymbol{H} = \frac{NI}{d} \boldsymbol{a}_z.$$

If the radius is r, corresponding to an area of $A = \pi r^2$, the self-inductance is

$$L = \frac{N\phi}{I} = \mu_0 N^2 \frac{A}{d}.$$

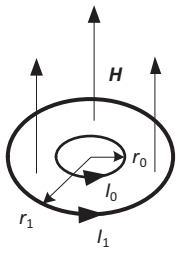

Figure 1.8 Two coaxial solenoids, top view

Now consider two coaxial solenoids, with radii r_1 and $r_0 < r_1$, carrying currents of I_1 and I_0, and with different numbers of turns N_1 and N_0, respectively. The top view is shown in Figure 1.8.

To find the mutual inductance M_{01}, we can write

$$\phi_{01} = \mu_0 A_0 H_0,$$

where $H_0 = \frac{N_0 I_0}{d}$ is the magnetic field intensity created by the smaller solenoid. Since H_0 is zero outside the radius of the smaller solenoid, we have

$$M_{01} = \frac{N_1}{I_0}\mu_0 A_0 H_0 = \mu_0 N_0 N_1 \frac{A_0}{d}.$$

A similar procedure leads to M_{10}, which comes out to be equal to M_{01}. This is in agreement with reciprocity, as expected.

1.3 TIME-VARYING FIELDS AND MAXWELL'S EQUATIONS

As described earlier, time-varying fields could also be sources of electric or magnetic field creation. In 1831, Faraday published his findings which resulted from performing the following experiment where he proved that a time-varying magnetic field does indeed result in a current. He wound two separate coils on an iron toroid, placed a galvanometer in one and a battery and a switch in the other (Figure 1.9). Upon closing the switch, he realized that the galvanometer

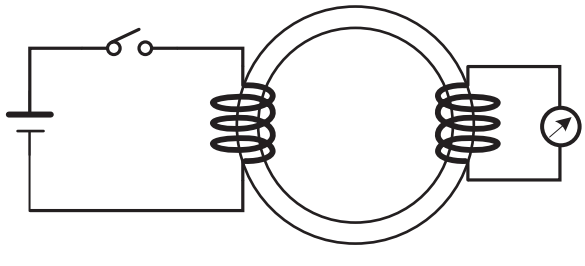

Figure 1.9 Faraday's experiment

was momentarily deflected. He observed the same deflection but in an opposite direction when the battery was disconnected. In terms of fields, we can say that a time-varying magnetic field (or flux) produces an *electromotive force* (emf, measured in volts) that may establish a current in a closed circuit. A time-varying magnetic field may be a result of a time-varying current, or the relative motion of a steady flux and a closed path, or a combination of the two.

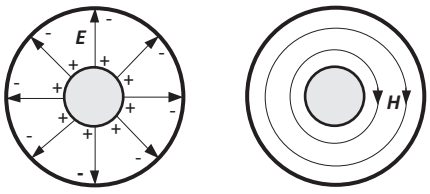

Figure 1.10 Lines of electric and magnetic fields in a coaxial cable

Faraday's law as stated above is customarily formulated as

$$\text{emf} = \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{L} = -\frac{\mathrm{d}\phi}{\mathrm{d}t},$$

where the line integral comes from the basic definition of voltage ($\boldsymbol{E}$ is the electric field intensity). The minus sign indicates that the emf is in such a direction as to produce a current whose flux, if added to the original one, would reduce the magnitude of the emf, and this is generally known as Lenz's law.

Similarly, a time-varying electric flux results in a magnetic field, and is generally formulated by modifying Ampère's circuital law as follows:

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{L} = I + \int_s \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{S},$$

where $\boldsymbol{D}$ is the electric flux density, and $\int_S \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}$ is termed the *displacement current* by Maxwell. To summarize, we can state the four Maxwell's equations in the integral form as follows:

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{L} = -\int_s \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S},$$

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{L} = I + \int_s \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{S},$$

$$\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho_V \mathrm{d}V,$$

$$\oint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0.$$

The third equation is Gauss's law, as discussed earlier. The fourth equation,[7] states that, unlike the electric fields that begin and terminate on positive and negative charges, the magnetic field forms *concentric circles*. In other words the magnetic flux lines are closed and do not terminate on a magnetic charge[8] (Figure 1.10). Therefore, the closed surface integral of a magnetic field (or magnetic flux density) is zero.

In free space where the medium is source-less, I (or ρ_V) is equal to zero. The first two of Maxwell's equations, when combined, lead to a differential equation relating the second-order

[7] The fourth equation is often known as Gauss's law for magnetism.
[8] Magnetic charges or monopoles are yet to be found in nature, although the magnetic monopole is used in physics as a *hypothetical* elementary particle.

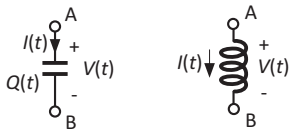

Figure 1.11 Capacitor and inductor circuit representation

derivative of $\boldsymbol{E}$ (or $\boldsymbol{H}$) versus space to its second-order derivative versus time, describing the *wave propagation* in free space. For example, if $\boldsymbol{E} = E_x \boldsymbol{a}_x$, or if electric field is *polarized* only in the x direction, with some straightforward math [6], and using Maxwell's equations in their differential form, it can be shown that[9]

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}.$$

The propagation is in the z direction, whose velocity is defined as

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c,$$

where $c = 3 \times 10^8$ m/s is the velocity of light in free space.

1.4 CIRCUIT REPRESENTATION OF CAPACITORS AND INDUCTORS

From a circuit point of view, a capacitor is symbolically represented as shown in Figure 1.11, and its voltage and current ($V(t)$ and $I(t)$) as shown satisfy the following relations [7]:

$$I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t},$$

where Q is the charge stored in the capacitor. The above expression is widely known as the *continuity equation*. For the case of a linear and time-invariant capacitor, since $Q = CV$, we can write the well-known expression for the capacitor:

$$I(t) = C\frac{\mathrm{d}V}{\mathrm{d}t}.$$

Note that the continuity equation as expressed in most physics books is $I(t) = -\frac{\mathrm{d}Q}{\mathrm{d}t}$, indicating that the *outward flow* of the positive charge must be balanced by a *decrease* of the charge within the closed surface (that is Q). The minus sign is omitted here, since in Figure 1.11 it is the inward current flow into one terminal of the capacitor with the time rate of increase of charge on that terminal, and not the *outward* current.

An inductor is symbolically represented as shown in Figure 1.11, where its voltage and current ($V(t)$ and $I(t)$) as shown satisfy the following relations:

$$V(t) = \frac{\mathrm{d}\phi}{\mathrm{d}t},$$

[9] The more general form of the wave equation is: $\nabla^2 \boldsymbol{E} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$.