

TURBULENCE AND SHELL MODELS

Turbulence is a huge subject of ongoing research. This book bridges modern developments in dynamical systems theory and the theory of fully developed turbulence. Many solved and unsolved problems in turbulence have equivalences in simple dynamical models, which are much easier to handle analytically and numerically.

This book gives a modern view of the subject by first giving the essentials of the theory of turbulence before moving on to shell models. These show much of the same complex behavior as fluid turbulence, but are much easier to handle analytically and numerically. Any necessary maths is explained and self-contained, making this book ideal for advanced undergraduates and graduate students, as well as for researchers and professionals, wanting to understand the basics of fully developed turbulence.

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Peter D. Ditlevsen
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Preface

Fluids have always fascinated scientists and their study goes back at least to the ancient Greeks. Archimedes gave in “On Floating Bodies” (c. 250 BC) a surprisingly accurate account of basic hydrostatics. In the fifteenth century, Leonardo da Vinci was an excellent observer and recorder of natural fluid flows, while Isaac Newton experimented with viscosity of different fluids reported in *Principia Mathematica* (1687); it was his mechanics that formed the basis for describing fluid flow. Daniel Bernoulli established his principle (of energy conservation) in a laminar inviscid flow in *Hydrodynamica* (1738). The mathematics of the governing equations was treated in the late eighteenth century by Euler, Lagrange, Laplace, and other mathematicians. By including viscosity the governing equations were put in their final form by Claude-Louis Navier (1822) and George Gabriel Stokes (1842) in the Navier–Stokes equation. This has been the basis for a vast body of research since then.

The engineering aspects range from understanding drag and lift in connection with design of airplanes, turbines, ships and so on to all kinds of fluid transports and pipeflows. In weather and climate predictions accurate numerical solutions of the governing equations are important. In all specific cases when the Reynolds number is high, turbulence develops and the kinetic energy is transferred to whirls and waves on smaller and smaller scales until eventually it is dissipated by viscosity. This is the energy cascade in turbulence. The difference in size between the scales where kinetic energy is inserted into the flow and the scales where it is dissipated as heat is huge. It ranges from, say, the whole atmosphere of the planet to the sub-millimeter scale where viscosity of the air is important.

This fundamental aspect of turbulence can be illuminated by shell models. Shell models have, through their simplicity, contributed to the understanding of symmetries, scaling and intermittency in turbulent systems. Their relatively low number of degrees of freedom in comparison to high Reynolds number flows has enabled them to bridge the gap between the chaotic dynamics observed in low

dimensional systems and turbulence. Their computational affordability and simplicity also make them ideal tools for students entering the field of turbulence, and for researchers to test ideas.

This book gives an introduction to the field of turbulence in the spirit of the Kolmogorov phenomenology represented by the famous “K41” scaling relation. The emphasis on shell models in their own right is that the governing equations for shell models share many aspects and are structurally similar to the Navier–Stokes equation, and they are just so much easier to handle. The book is intended for researchers and professionals who want a fast introduction to the problem of isotropic and homogeneous turbulence in the spirit of dynamical systems theory. It should be accessible for advanced undergraduate and graduate students. Most of the material has been used for teaching the subject at the graduate level. For that, a set of problems can be found at the end of each chapter. There are two types of problem: some address the concepts, the mathematics or completion of the calculations leading to the results in the text. Other problems introduce concepts or phenomena, such as Burgers equation, not treated in the main text. An asterisk * indicates a difficult exercise. Shell models are perfect “lab-systems” for numerical investigations, both for testing new scientific ideas and for students to reproduce theoretical results and to familiarize with concepts like scaling relations, Lyapunov exponents and intermittency. To maintain the flow of the main text, some of the mathematical and technical details are deferred to appendices.