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Introduction to turbulence

Fully developed turbulence is the notion of the general or universal behavior in any physical situation of a violent fluid flow, be it a dust devil or a cyclone in the atmosphere, the water flow in a white-water river, the rapid mixing of the cream and the coffee when stirring in a coffee cup, or perhaps even the flow in gigantic interstellar gas clouds. It is generally believed that the developments of these different phenomena are describable through the Navier–Stokes equation with suitable initial or boundary conditions. The governing equation has been known for almost two centuries, and a lot of progress has been achieved within practical engineering in fields like aerodynamics, hydrology, and weather forecasting with the ability to perform extensive numerical calculations on computers. However, there are still fundamental questions concerning the nature of fully developed turbulence which have not been answered. This is perhaps the biggest challenge in classical physics. The literature on the subject is vast and very few people, if any, have a full overview of the subject. In the updated version of Monin and Yaglom's classic book the bibliography alone covers more than 60 pages (Monin & Yaglom, 1981).

The phenomenology of turbulence was described by Richardson (1922) and quantified in a scaling theory by Kolmogorov (1941b). This description stands today, and has been shown to be basically correct by numerous experiments and observations. However, there are corrections which are not explainable by the Kolmogorov theory. These corrections are deviations in scaling exponents for the scaling of correlation functions. The Kolmogorov theory is not based on the Navier–Stokes equation, except for one of the very few exact relations, namely the four-fifth law, describing the scaling of a third order correlation function. A final theory explaining the corrections should be based on the Navier–Stokes equation.

Shell models of turbulence were introduced by Obukhov (1971) and Gledzer (1973). They consist of a set of ordinary differential equations structurally similar to the spectral Navier–Stokes equation. These models are much simpler and numerically easier to investigate than the Navier–Stokes equation. For these models a scaling theory identical to the Kolmogorov theory has been developed, and they show the same kind of deviation from the Kolmogorov scaling as real turbulent systems do. Understanding the behavior of shell models in their own right might be a key for understanding the systems governed by the Navier–Stokes equation. The shell models are constructed to obey the same conservation laws and symmetries as the Navier–Stokes equation. As well as energy conservation, the models exhibit, depending on a free parameter of the model, conservation of a second quantity which can be identified with helicity or enstrophy. This second quantity signifies whether the models are 3D turbulence-like where helicity is conserved, or 2D turbulence-like where enstrophy is conserved.

In the case of 3D helical (non-mirror symmetric) turbulent flow, there exists a dual cascade of energy and helicity to small scales. A wave number in the inertial range smaller than the Kolmogorov wave number, where the helicity dissipation becomes important, has been identified. In the case of shell models the flow becomes non-helical from this wave number on, until energy is dissipated around the Kolmogorov wave number. In the case of 2D turbulence a forward cascade of energy to small scales is prohibited altogether by the cascade of enstrophy. On the contrary, the energy is transported upscale in an inverse cascade. In the case of shell models we can investigate under what circumstances this is compatible with equipartitioning of the conserved quantities in a quasi-equilibrium as predicted by equilibrium statistical mechanics.

The corrections to the Kolmogorov scaling expressed through the anomalous scaling exponents can be qualitatively understood as a consequence of intermittency in the energy cascade. By simulation we can make a qualitative link between the multi-fractal cascade models and the shell models. The relative simplicity of the shell models also makes it possible to describe the dynamics in terms of bifurcations, routes to chaos, Lyapunov exponents, and so on using the tools developed in the theory of chaos for low dimensional dynamical systems.

This chapter presents a review of some of the main characteristics and unknowns of turbulence. Turbulence is the chaotic and apparently random flow of a stirred fluid. Fluid flows vary a lot depending on the boundaries containing the flow, stirring, and heating. The flow in the atmosphere of a rotating planet is different from the convection in a pot of boiling water. However, as long as the length scales in the flow are small in comparison to the largest scales, determined by the boundaries, and large in comparison to the molecular mean free path scales, all flows

seem to have common characteristics. Turbulence is this common characteristic of the flows.

1.1 The Navier–Stokes equation

Fluid mechanics is the description of fluids on scales large in comparison to the mean free path length of the molecules constituting the fluid. In this limit the fluid is regarded as a continuum characterized completely by a velocity field $u_i(\mathbf{x},t)$, a temperature field $T(\mathbf{x})$, a pressure field $p(\mathbf{x})$ and a density field $\rho(\mathbf{x})$. At each point x_i the fluid is then fully characterized by the six field variables: three components of velocity, pressure, temperature, and density. In order to determine the evolution of these we need six equations. These are derived from momentum conservation, mass conservation, energy conservation, and the equation of state. In any concrete setting some of the field variables might be approximately constant and the number of equations reduced. When considering fully developed turbulence the fluid is traditionally regarded as incompressible, which is a rather good approximation for water and air. This immediately eliminates the equation specifying density from consideration. When buoyancy can be neglected the temperature variations decouple from the momentum and continuity equations and we are left with a fluid described by the velocity and the pressure field. The dynamics of such a fluid is described by the Navier–Stokes equation (NSE)

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \nu \partial_{jj} u_i + f_i, \tag{1.1}$$

and the continuity equation

$$\partial_i u_i = 0. \tag{1.2}$$

The NSE describes the conservation of momentum. In this book we mainly use the tensor notation: $\partial_i u_j \equiv \partial u_j/\partial x_i$, $\partial_{ij} u_k \equiv \partial^2 u_k/\partial x_i \partial x_j$, etc., and the Einstein convention of summing repeated indices; $\partial_{kk} f \equiv \Delta f$ denotes the Laplacian of f. The equation states that the acceleration of a fluid particle equals the sum of the forces acting on the fluid particle (per unit mass). The left hand side is the material derivative of the velocity field, where the second term is the advection. The first term on the right hand side is the pressure gradient force, the second is the viscous friction (viscosity) and the last term represents all other forces per unit of mass. The last term is, historically speaking, not a part of the NSE but we will specify it whenever convenient.

The continuity equation is the equation for conservation of mass, where in the case of an incompressible fluid the density does not appear. The inverse of the density, which normally appears in front of the gradient of pressure in (1.1) is thus also absorbed in the units of pressure. From these (four) equations, together with

boundary and initial conditions, the three components of the fluid velocity u_i and the pressure p can in principle be determined.

However, no general solutions to the NSE are known and a solution can be found only for very simple laminar flows. Pressure can immediately be eliminated from the NSE using the continuity Equation (1.2). Assuming the force to be rotational, $\partial_i f_i = 0$, we obtain a Poisson equation for the pressure by applying the divergence operator to the NSE

$$\partial_{ii} p = -\partial_i u_j \partial_j u_i, \tag{1.3}$$

which we can formally solve by applying the inverse Laplacian

$$p = -\partial_{kk}^{-1}(\partial_i u_j \partial_j u_i). \tag{1.4}$$

We make sense of the inverse of a differential operator when expressing the NSE in terms of Fourier components.

The NSE can be brought to a dimensionless form by defining

$$x = L\tilde{x},\ u = U\tilde{u},\ t = (L/U)\tilde{t}, \tag{1.5}$$

where L is the length scale of the largest variations in the flow. Note that L would typically be the size of the container or basin for a bounded flow or the size of an obstacle in an extended uniform flow; L is called the integral or outer scale; U is the typical velocity difference at this length scale. We can think of U as the typical velocity when coarse graining the flow at the length scale L. As it is derived from Newton's second law the NSE is Galilean invariant. This means that adding a uniform velocity, say, by moving the frame of reference, does not change the NSE. Thus the overall uniform center of mass velocity is unchanged (in the case that the sum of external forces vanishes) and only velocity differences are important. From L and U we can build a timescale $T = L/U$, which is just the time it takes the fluid at uniform velocity U to travel the distance L. Inserting this into (1.1) and dropping the tilde "˜" gives the NSE in dimensionless form:

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \mathrm{Re}^{-1} \partial_{jj} u_i + f_i, \tag{1.6}$$

where we have defined the dimensionless Reynolds number

$$\mathrm{Re} \equiv \frac{UL}{\nu}, \tag{1.7}$$

and absorbed a factor U^2/L into the forcing term. The pressure gradient term is, as can be seen from (1.4), dimensionally the gradient of a velocity squared, so that it scales with changing units of length and time like the advection term. All terms except for the viscosity are now of order unity. The viscosity is of the order of

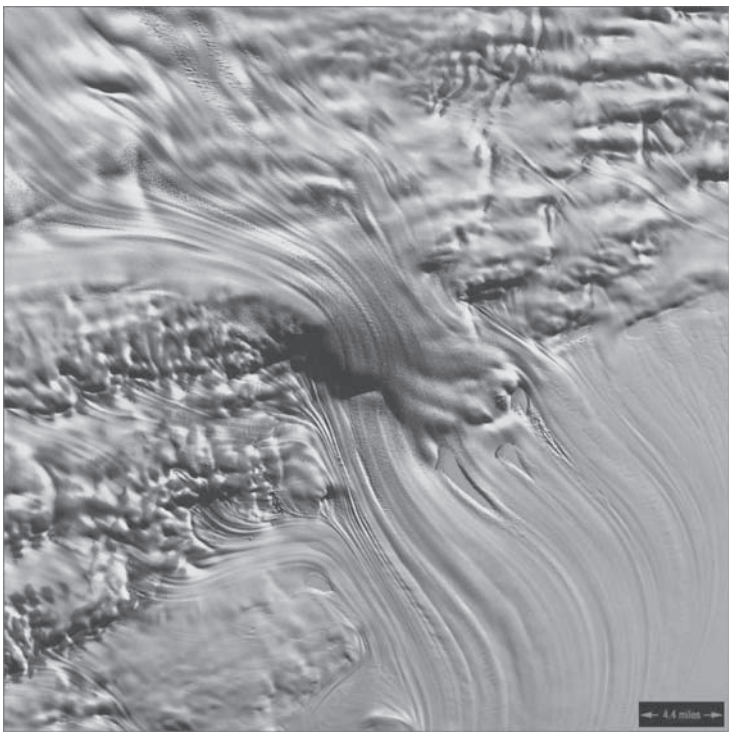

Figure 1.1 Low Reynolds number flow of ice where the viscosity completely dominates the flow. The pheomenon is called an icefall. The fall is about 400 m at the Lambert Glacier, Antarctica (NASA/Landsat).

the inverse of the Reynolds number, so that the Reynolds number is a measure of the relative importance of the viscosity in comparison to the nonlinear terms (the advection and the pressure gradient term) at the length scale L and velocity scale U. The Reynolds number is the fundamental characteristic of any given flow. For a Reynolds number smaller than one the flow will quickly be damped by viscosity, or the viscous term will balance the external forces such as gravity, as is the case in Figure 1.1. The viscosity acts as a smoother of irregularities and has the form of a diffusion term. When the Reynolds number becomes larger the flow will be more and more dominated by the nonlinear terms.

For small Reynolds numbers the flow is smooth and regular. As the Reynolds number is increased the fluid motion in the wake becomes more and more irregular. Increasing Reynolds number flow can be seen as a successive symmetry breaking. For very high Reynolds numbers the regularity of the von Kármán street shown in Figure 1.2 disappears and the flow is completely chaotic and apparently random. This is called *fully developed turbulence*. It characterizes many systems in nature,

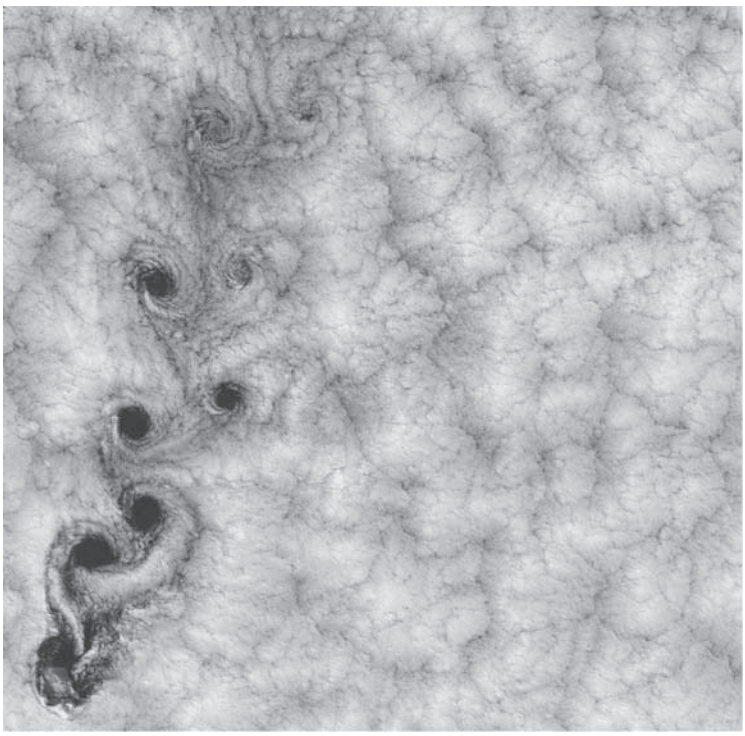

Figure 1.2 Atmospheric flow around Selkirk Island in the southern Pacific Ocean, where the dense cloud cover makes the flow visible. The highest point of the island is about 1.6 km above sea level, obstructing the flow. The phenomenon is called a von Kármán vortex street. (NASA/Landsat)

such as the flow in the atmospheric boundary layer, river flow, the wake after a jet-engine, smoke from a cigar, and many other phenomena. All the richness of the complex behavior of these systems is, we believe, described by the NSE. Direct numerical simulations of the NSE indeed show some of this richness. However, no general theory exists with which we can relate directly the NSE and the rich phenomenology observed in nature and experiments. For high Reynolds number flow there will be a large range of scales where the viscous dissipation is negligible. Assuming either a non-forced decaying motion or forcing restricted to the large scales, motion in this range will be determined by inertia. This is thus called the inertial range. Fully developed turbulence is characterized by a long inertial range. The structure and dynamics of different flows in this range seem in some statistical sense to be alike and one may ask if there is some universality in the behavior of the flows.

The common phenomenology of fully developed turbulence is attributed to Richardson (1922). Richardson describes the flow as consisting of large swirls

breaking up into smaller swirls, which again break up into yet smaller swirls until finally the swirls are so small that they are smoothed out, or dissipated, by the viscosity. The energy is inserted into the flow at large scales, it then cascades into smaller and smaller scales until it leaves the flow at the viscous scale. In 1941 this led Kolmogorov to develop a phenomenological theory of turbulence.

1.2 Kolmogorov's 1941 theory (K41)

The paper in which Kolmogorov (1941b) presents the theory is only one half page long and the idea is very simple. It is presented thoroughly by Landau and Lifshitz (1987). Here we will go through it briefly. Kolmogorov imagined a flow initiated by vigorous stirring and then left alone to slowly dampen out by viscosity. This case of unforced flow is today called decaying turbulence. The flow is assumed to be homogeneous (translationally invariant in the mean) and isotropic (rotationally invariant in the mean). The picture we have in mind here is a flow maintained by a force active on large scales of the flow, such that the flow is in a state of statistical equilibrium in the sense that on average the energy input by the force is balanced by the energy dissipated by viscosity (heating the fluid). The state of the flow is then characterized by the mean energy dissipation per unit of mass $\overline{\varepsilon}$ due to viscosity. The velocity characteristic of a given length scale $l \ll L$ is the typical velocity difference $\delta u(l) \equiv |u(r+l) - u(r)|$, where for clarity we suppress all vector indices. This velocity difference is characteristic of the velocity associated with an eddy of size l. The effect of the larger scale flow velocity is merely to move, or sweep, the eddy through the flow as a rigid body. Likewise, if we consider a much smaller eddy within the larger eddy, the effect of the larger eddy on the smaller is the same as the effect of the larger scale flow on the large eddy. Since there is nothing physically significant about a given length scale l in the flow we assume the flow to be self-similar in the sense that when $l_1 < l_2 \ll L$ the velocity differences are related by $\delta u(l_2) = f(l_1/l_2)\delta u(l_1)$, where f is some universal function. This implies that the velocity difference $\delta u(l)$ can only be a function of the scale l and the mean energy dissipation $\overline{\varepsilon}$. From dimensional counting the only possible relationship is

$$\delta u(l) \sim (\overline{\varepsilon}\, l)^{1/3}, \tag{1.8}$$

where $\sim$ means proportionality. The eddy turnover time is the typical timescale for a fluid parcel propagating across the size of the eddy with the typical velocity associated with that eddy.

We use these kinds of dimensional argument throughout this book. The scaling relation (1.8) is obtained from the fact that only quantities of the same

dimension can be compared. So if we want to establish a functional relationship

$$\delta u(l) = \tilde{f}(l, \overline{\varepsilon}), \tag{1.9}$$

the dimension of the right hand side must be the same as the dimension of the left hand side. Furthermore, the numerical value of the quantity on the left hand side cannot depend on change of units of the quantities on the right hand side which leave the units on the left hand side unchanged. If, for example, we measure length in millimeters and time in milliseconds instead of metres and seconds, the numerical value of velocity is unchanged. However, the numerical values of l and $\overline{\varepsilon}$ change. If (1.9) is to hold regardless of the change of units, f can only depend on the combination of l and $\overline{\varepsilon}$ which has the same dimension as the left hand side. The dimensions are $[\delta u]$ =m/s, $[l]$ = m, $[\overline{\varepsilon}] = \mathrm{m}^2/\mathrm{s}^3$, so for $[\delta u] = [l]^\alpha [\overline{\varepsilon}]^\beta$ we get, $\beta = \alpha = 1/3$. From this we get $\delta u(l) = f[(\overline{\varepsilon} l)^{1/3}]$. By changing the units of velocity, say scaling length by a factor λ, we get

$$\lambda\, \delta u(l) = \lambda f[(\overline{\varepsilon}\, l)^{1/3}] = f[\lambda (\overline{\varepsilon}\, l)^{1/3}]. \tag{1.10}$$

Thus we see that f must be a linear function and we obtain (1.8).

The relation (1.8) contains all the essentials of the K41 theory. The scale η at which the dissipation becomes important is called the Kolmogorov, or inner scale, in contrast to L, the outer, or integral scale. From (1.1) we can get an estimate of the rate of change of the energy per unit volume due to dissipation at the scale η, $\overline{\varepsilon} \sim \nu u_i \partial_{jj} u_i \sim \nu \delta u(\eta)^2/\eta^2$. Using (1.8) we get

$$\eta \sim (\overline{\varepsilon}/\nu^3)^{-1/4}. \tag{1.11}$$

So keeping the integral length scale velocity and the mean energy dissipation $\overline{\varepsilon}$ fixed, the Kolmogorov scale depends on the Reynolds number as $\eta \sim \mathrm{R}e^{-3/4}$.

The mean of the square of the velocity difference is called the second order structure function $S_2(l)$. The scaling of $S_2(l)$ is obtained by simply squaring (1.8):

$$S_2(l) \equiv \langle \delta u(l)^2 \rangle \sim (\overline{\varepsilon} l)^{2/3}. \tag{1.12}$$

The mean $\langle . \rangle$ denotes an ensemble average, defined as the average over many realizations of the flow with different initial conditions (drawn from some distribution). Assuming ergodicity, this could as well be a temporal average (in a given set of points), or a spatial average in the case of homogeneity. We will freely assume these three to be equal or use either at our convenience without dwelling more on subtleties regarding the assumption of ergodicity or the distribution of initial conditions. A rigorous discussion can be found in Frisch (1995).

1.3 The spectral Navier–Stokes equation

Consider the Fourier transform of the velocity field, Equations (A.3) and (A.4):

$$\mathcal{F}_-: \quad u_i(\mathbf{k}) = \frac{1}{(2\pi)^3}\int e^{-\iota \mathbf{kx}} u_i(\mathbf{x})\mathrm{d}\mathbf{x}, \tag{1.13}$$

$$\mathcal{F}_+: \quad u_i(\mathbf{x}) = \int e^{\iota \mathbf{kx}} u_i(\mathbf{k})\mathrm{d}\mathbf{k}, \tag{1.14}$$

using the notation introduced in Appendix A.1.

Transforming by $\mathcal{F}_-$, using (A.9)–(A.11), the NSE (1.1), and the Poisson Equation (1.3) give,

$$\begin{aligned}\partial_t u_i(\mathbf{k}) = -\iota \int u_j(\mathbf{k}-\mathbf{k}')\, k'_j\, u_i(\mathbf{k}')\mathrm{d}\mathbf{k}' \\ -\iota k_i p(\mathbf{k}) - \nu k_j k_j u_i(\mathbf{k}) + f_i(\mathbf{k}),\end{aligned} \tag{1.15}$$

and

$$\begin{aligned}-k_j k_j p(\mathbf{k}) &= -\int (k_i - k'_i)\, u_j(\mathbf{k}-\mathbf{k}')\, k'_l\, u_m(\mathbf{k}')\mathrm{d}\mathbf{k}'\delta_{lj}\delta_{mi} \\ &= -\int (k_j - k'_j)\, u_l(\mathbf{k}-\mathbf{k}')\, k'_l\, u_j(\mathbf{k}')\mathrm{d}\mathbf{k}' \\ &= -\int k_j k'_l\, u_l(\mathbf{k}-\mathbf{k}')\, u_j(\mathbf{k}')\mathrm{d}\mathbf{k}'.\end{aligned} \tag{1.16}$$

It has been noted that incompressibility implies $k'_j u_j(\mathbf{k}') = 0$. Substitution of $p(\mathbf{k})$ from (1.16) into (1.15) gives the spectral NSE

$$\begin{aligned}\partial_t u_i(\mathbf{k}) = -\iota k_j \int \left(\delta_{il} - \frac{k_i k'_l}{k^2}\right) u_j(\mathbf{k}')u_l(\mathbf{k}-\mathbf{k}')\mathrm{d}\mathbf{k}' \\ -\nu k^2 u_i(\mathbf{k}) + f_i(\mathbf{k}),\end{aligned} \tag{1.17}$$

where $k^2 = k_j k_j$.

If we consider the flow in a box of size L^3 with periodic boundary conditions, the Fourier transform is substituted by a Fourier series and the integral in (1.17) becomes a sum

$$\partial_t u_i(\mathbf{n}) = -\iota\,(2\pi/L)\,n_j \sum_{\mathbf{n}'} \left(\delta_{il} - \frac{n_i n_l'}{n^2}\right) u_j(\mathbf{n}')u_l(\mathbf{n}-\mathbf{n}') - \nu n^2 u_i(\mathbf{n}) + f_i(\mathbf{n}), \tag{1.18}$$

where the wave vectors are $\mathbf{k}(\mathbf{n}) = 2\pi\mathbf{n}/L$, and $n^2 = n_j n_j$. This form of the NSE is the starting point for the shell models. The partial differential equation (PDE) (1.1) has now been substituted by a hierarchy of coupled ordinary differential equations (ODEs). The nonlinear terms are quadratic in the velocities. The interactions are such that only waves with wave vectors adding up to zero are coupled. Such a set of three waves is called a triad. It can be shown by manipulating indices in (1.17) and (1.18) that the inviscid energy conservation fulfilled by the NSE is a detailed energy balance, so that energy is exchanged within each triad. The algebra involved in proving this and many other relations of the NSE is much simpler, but completely similar in the case of shell models. We thus for transparency do many of the calculations for the case of shell models.

1.4 The spectral energy density

The second order structure function (1.12) is related to the spectral energy density through a Fourier transform, as we now show. The energy density per unit of mass of the flow can be expressed in spectral form by use of Parseval's identity (A.7):

$$E = \frac{1}{2}\int \mathbf{u}(\mathbf{x})^2 \mathrm{d}\mathbf{x} = \frac{1}{2}(2\pi)^3 \int_0^\infty u_i(\mathbf{k})u_i(\mathbf{k})^* \mathrm{d}\mathbf{k} = \frac{1}{2}(2\pi)^3 4\pi \int_0^\infty k^2 |\mathbf{u}(k)|^2 \mathrm{d}k \equiv \int E(k)\mathrm{d}k, \tag{1.19}$$

where we have assumed the flow to be isotropic, $u_i(\mathbf{k}) = u_i(k)$, and performed the integration over the sphere. We have absorbed the unit of length into the spatial variable $\mathrm{d}\mathbf{x} \to \mathrm{d}\mathbf{x}/L^3$, where L is the linear size of the integration box. Thus we define the spectral energy density as

$$E(k) = 2\pi(2\pi)^3 k^2 |\mathbf{u}(k)|^2. \tag{1.20}$$

The Fourier transform of the velocity is expressed in terms of the second order structure function

$$S_2(\mathbf{l}) = \langle \delta\mathbf{u}(\mathbf{l})^2 \rangle = \int [\mathbf{u}(\mathbf{l}+\mathbf{x}) - \mathbf{u}(\mathbf{x})]^2 \mathrm{d}\mathbf{x} = 2\int [\mathbf{u}(\mathbf{x})^2 - \mathbf{u}(\mathbf{l}+\mathbf{x})\mathbf{u}(\mathbf{x})]\mathrm{d}\mathbf{x}. \tag{1.21}$$