ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Noncommutative Rational Series with Applications

JEAN BERSTEL
Université Paris-Est

CHRISTOPHE REUTENAUER
University of Québec, Montréal

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521190220

First published 2011

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging in Publication data
Berstel, Jean, 1941–
Noncommutative rational series with applications / Jean Berstel, Christophe Reutenauer.
p. cm. – (Encyclopedia of mathematics and its applications ; 137)
Includes bibliographical references and index.
ISBN 978-0-521-19022-0 (hardback)
1. Machine theory. 2. Noncommutative algebras. I. Reutenauer, Christophe.
II. Title. III. Series.
QA267.B47 2010
511.3´5 – dc22 2010030398

ISBN 978-0-521-19022-0 Hardback

Pour Anne et Anissa

Contents

Part III Applications

Preface

Formal power series have long been used in all branches of mathematics. They are invaluable in algebra, analysis, combinatorics and in theoretical computer science.

Historically, the work of M.-P. Schützenberger in the algebraic theory of finite automata and the corresponding languages has led him to introduce noncommutative formal power series. This appears in particular in his work with Chomsky on formal grammars. This last point of view is at the origin of this book.

The first part of the book, composed of Chapters 1–4, is especially devoted to this aspect: Formal power series may be viewed as formal languages with coefficients, and finite automata (and more generally weighted automata) may be considered as linear representations of the free monoid. In this sense, via formal power series, algebraic theory of automata becomes a part of representation theory.

The first two chapters contain general results and discuss in particular the equality between rational and recognizable series (Theorem of Kleene–Schützenberger) and the construction of the minimal linear representation. The exposition illustrates the synthesis of linear algebra and syntactic methods inherited from automata theory.

The next two chapters are concerned with the comparison of some typical properties of rational (regular) languages, when they are transposed to rational series. First, Chapter 3 describes the relationship with the family of regular languages studied in theoretical computer science. Next, the chapter contains iteration properties for rational series, also known as pumping lemmas, which are much more involved than those for regular languages. Chapter 4 discusses rational expressions. It contains two main results: the so-called "triviality" of rational identities over a commutative ring and the characterization of the star height of a rational series and its two consequences: the star height is unbounded and the star height over an algebraically closed field is decidable. The same problem for rational languages is known to be extremely difficult.

The second part of the book, composed of Chapters 5–8, is devoted to arithmetic properties of rational series.

Chapter 5 is concerned with automatic sequences and algebraic series. Two main results are the characterization of algebraic series over a finite field by Christol, Kamae, Mendès France and Rauzy, and Fürstenberg's theorem on the diagonal of a rational function.

Chapter 6 gives the proof of a theorem of Pólya characterizing rational series whose set of coefficients have only finitely many distinct prime divisors, and an elementary proof of a theorem of Skolem, Mahler and Lech about vanishing terms in a rational series.

Chapter 7 studies rational series over a principal ring and Fatou extensions. It contains a section on polynomial identities and rationality criteria, and a section on Fatou rings.

Chapter 8 is concerned with rational series with nonnegative coefficients. It contains a simplified proof of Soittola's theorem (Theorem 8.3.1) which is one of the most striking results on these rational series in one variable. Also the presentation of the star height (Theorem 8.4.1) is new.

The third part, composed of the remaining four chapters, is concerned with applications and with the study of important subfamilies of rational series.

Chapter 9 contains some results appearing for the first time in book form. The first section is on the Burnside problem of matrix semigroups. Section 9.2 contains a main result: Schützenberger's theorem on polynomially bounded rational series, one of the most difficult results in the area. The chapter also contains Simon's result on the Burnside problem for the matrix semigroups over the tropical semiring and limitedness of languages.

The next two chapters are devoted to the study of polynomials in noncommutative variables, and to their application to coding theory. Because of noncommutativity, the structure of polynomials is much more complex than in the case of commutativity, and the results are rather delicate to prove. We present here basic properties concerning factorizations. The main purpose of Chapter 10 is to prepare the ground for Chapter 11. The latter contains the generalization of a result of M.-P. Schützenberger concerning the factorization of a polynomial associated with a finite code.

Chapter 12 is a step towards representation theory. It gives results on semisimple syntactic algebras. Main results are the semisimplicity of the syntactic algebra of bifix codes and its converse. The syntactic algebra of a cyclic language is semisimple and its zeta function is rational. The chapter also contains a long appendix on the Rees–Suschkewitsch theorem which describes the structure of the minimal idea of a finite monoid. We included a self-contained exposition for the ease of the reader.

More than 170 exercises are provided, and also short bibliographical notes are given at the end of the chapters.

This book is issued from a previous book of the authors, entitled *Rational Series and their Languages*. The present text is an entirely rewritten and enriched version of this book. An important part of the material presented here appears for the first time in book form.

The text served for advanced courses held several times by the authors, at the University Pierre et Marie Curie, Paris and at the University of Saarbrücken. Parts of the book were also taught at several different levels at other places, such as the University of Québec at Montréal and the University of Marne-la-Vallée.

Many thanks to Sylvain Lavallée, Aaron Lauve, Martin Dagenais, Pierre Bouchard, Franco Saliola, Dominique Perrin and to Christian Mathissen for their help.

Note to the reader

Following usual notation, items such as sections, theorems, corollaries, etc. are numbered within a chapter. When cross-referenced the chapter number is omitted if the item is within the current chapter. Thus “Theorem 1.1” means the first theorem in the first section of the current chapter, and “Theorem 2.1.1” refers to the equivalent theorem in Chapter 2. Exercises are numbered accordingly and the section number should help the reader to find the section relevant to that exercise.