

1

Association schemes

1.1 Partitions

Association schemes are about relations between pairs of elements of a set Ω . In this book Ω will always be finite. Recall that $\Omega \times \Omega$ is the set of ordered pairs of elements of Ω ; that is,

$$\Omega \times \Omega = \{(\alpha, \beta) : \alpha \in \Omega, \beta \in \Omega\}.$$

I shall give three equivalent definitions of association scheme, in terms of partitions, graphs and matrices respectively. Each definition has advantages and disadvantages, depending on the context.

Recall that a *partition* of a set Δ is a set of non-empty subsets of Δ which are mutually disjoint and whose union is Δ .

Let \mathcal{C} be any subset of $\Omega \times \Omega$. Its *dual subset* is \mathcal{C}' , where

$$\mathcal{C}' = \{(\beta, \alpha) : (\alpha, \beta) \in \mathcal{C}\}.$$

We say that \mathcal{C} is *symmetric* if $\mathcal{C} = \mathcal{C}'$. One special symmetric subset is the *diagonal* subset $\text{Diag}(\Omega)$ defined by

$$\text{Diag}(\Omega) = \{(\omega, \omega) : \omega \in \Omega\}.$$

First definition of association scheme An *association scheme* with s associate classes on a finite set Ω is a partition of $\Omega \times \Omega$ into sets $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_s$ (called *associate classes*) such that

- (i) $\mathcal{C}_0 = \text{Diag}(\Omega)$;
- (ii) \mathcal{C}_i is symmetric for $i = 1, \dots, s$;
- (iii) for all i, j, k in $\{0, \dots, s\}$ there is an integer p_{ij}^k such that, for all (α, β) in \mathcal{C}_k ,

$$|\{\gamma \in \Omega : (\alpha, \gamma) \in \mathcal{C}_i \text{ and } (\gamma, \beta) \in \mathcal{C}_j\}| = p_{ij}^k.$$

Note that the superscript k in p_{ij}^k does not signify a power.

Elements α and β of Ω are called i -th associates if $(\alpha, \beta) \in \mathcal{C}_i$. The number $s + 1$, which is the true number of associate classes, is called the *rank* of the association scheme.

We can visualize $\Omega \times \Omega$ as a square array whose rows and columns are indexed by Ω , as in Figure 1.1. If the rows and columns are indexed in the same order, then the diagonal subset consists of the elements on the main diagonal, which are marked 0 in Figure 1.1. Condition (i) says that \mathcal{C}_0 is precisely this diagonal subset. Condition (ii) says that every other associate class is symmetric about that diagonal: if the whole picture is reflected about that diagonal then the associate classes remain the same. For example, the set of elements marked * could form an associate class if symmetry were the only requirement. Condition (iii) is much harder to visualize for partitions, but is easier to interpret in the later two definitions.

| | | | | | | |
|----------|---|----------|---|---|---|---|
| | | Ω | | | | |
| | 0 | * | | | * | |
| | * | 0 | | | | |
| | | | 0 | | | * |
| Ω | | | 0 | | | |
| | * | | | 0 | | |
| | | | | | 0 | |
| | | * | | | | 0 |

Fig. 1.1. The elements of $\Omega \times \Omega$

Note that condition (ii) implies that $p_{ij}^0 = 0$ if $i \neq j$. Similarly, $p_{0j}^k = 0$ if $j \neq k$ and $p_{i0}^k = 0$ if $i \neq k$, while $p_{0j}^j = 1 = p_{i0}^i$. Condition (iii) implies that every element of Ω has p_{ii}^0 i -th associates, so that in fact the set of elements marked * in Figure 1.1 could not be an associate class. Write $a_i = p_{ii}^0$. This is called the *valency* of the i -th associate class.

(Many authors use k_i or n_i to denote valency, but this conflicts with the very natural use of n_i in Chapters 3 and 6 and k in Chapters 4–7.)

The integers $|\Omega|$, s , a_i for $0 \leq i \leq s$ and p_{ij}^k for $0 \leq i, j, k \leq s$ are called the *parameters of the first kind*. Note that $p_{ij}^k = p_{ji}^k$.

Example 1.1 Let $\Delta_1 \cup \dots \cup \Delta_b$ be a partition of Ω into b subsets of size m , where $b \geq 2$ and $m \geq 2$. These subsets are traditionally called *groups*, even though they have nothing to do with the algebraic structure called a group. Let α and β be

- first associates if they are in the same group but $\alpha \neq \beta$;
- second associates if they are in different groups.

See Figure 1.2.

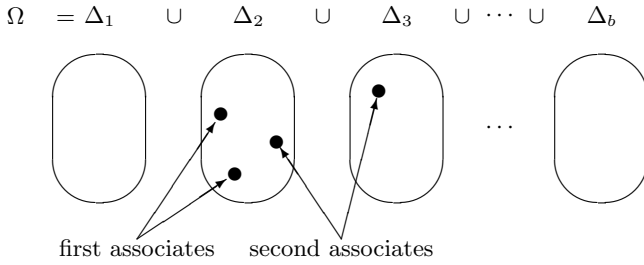


Fig. 1.2. Partition of Ω in Example 1.1

If $\omega \in \Omega$ then ω has $m - 1$ first associates and $(b - 1)m$ second associates.

If α and β are first associates then the number of γ which are specified associates of α and β are:

| | | |
|------------------------------|----------------------------|-----------------------------|
| | first associate of β | second associate of β |
| first associate of α | $m - 2$ | 0 |
| second associate of α | 0 | $(b - 1)m$ |

For example, those elements which are first associates of both α and β are the $m - 2$ other elements in the group which contains α and β . If α and β are second associates then the number of γ which are specified associates of α and β are:

| | | |
|------------------------------|----------------------------|-----------------------------|
| | first associate of β | second associate of β |
| first associate of α | 0 | $m - 1$ |
| second associate of α | $m - 1$ | $(b - 2)m$ |

So this is an association scheme with $s = 2$, $a_1 = m - 1$, $a_2 = (b - 1)m$, and

$$\begin{aligned}
 p_{11}^1 &= m - 2 & p_{12}^1 &= 0 & p_{11}^2 &= 0 & p_{12}^2 &= m - 1 \\
 p_{21}^1 &= 0 & p_{22}^1 &= (b - 1)m & p_{21}^2 &= m - 1 & p_{22}^2 &= (b - 2)m
 \end{aligned}$$

It is called the *group-divisible* association scheme, denoted $\text{GD}(b, m)$ or $\underline{b/m}$. ■

(The name ‘group-divisible’ has stuck, because that is the name originally used by Bose and Nair. But anyone who uses the word ‘group’ in its algebraic sense is upset by this. It seems to me to be quite acceptable to call the scheme just ‘divisible’. Some authors tried to compromise by calling it ‘groop-divisible’, but the nonce word ‘groop’ has not found wide approval.)

To save writing phrases like ‘first associates of α ’ we introduce the notation $\mathcal{C}_i(\alpha)$ for the set of i -th associates of α . That is,

$$\mathcal{C}_i(\alpha) = \{\beta \in \Omega : (\alpha, \beta) \in \mathcal{C}_i\}.$$

Thus condition (iii) says that $|\mathcal{C}_i(\alpha) \cap \mathcal{C}_j(\beta)| = p_{ij}^k$ if $\beta \in \mathcal{C}_k(\alpha)$. For this reason, the parameters p_{ij}^k are called the *intersection numbers* of the association scheme.

Example 1.2 Let $|\Omega| = n$, let \mathcal{C}_0 be the diagonal subset and let

$$\mathcal{C}_1 = \{(\alpha, \beta) \in \Omega \times \Omega : \alpha \neq \beta\} = (\Omega \times \Omega) \setminus \mathcal{C}_0.$$

This is the *trivial* association scheme – the only association scheme on Ω with only one associate class. It has $a_1 = n - 1$ and $p_{11}^1 = n - 2$. I shall denote it \underline{n} . ■

Example 1.3 Let Ω be an $n \times m$ rectangular array with $n \geq 2$ and $m \geq 2$, as in Figure 1.3. Note that this is a picture of Ω itself, not of $\Omega \times \Omega$! Put

$$\begin{aligned} \mathcal{C}_1 &= \{(\alpha, \beta) : \alpha, \beta \text{ are in the same row but } \alpha \neq \beta\} \\ \mathcal{C}_2 &= \{(\alpha, \beta) : \alpha, \beta \text{ are in the same column but } \alpha \neq \beta\} \\ \mathcal{C}_3 &= \{(\alpha, \beta) : \alpha, \beta \text{ are in different rows and columns}\} \\ &= (\Omega \times \Omega) \setminus \mathcal{C}_0 \setminus \mathcal{C}_1 \setminus \mathcal{C}_2. \end{aligned}$$

Then $a_1 = m - 1$, $a_2 = n - 1$ and $a_3 = (m - 1)(n - 1)$.

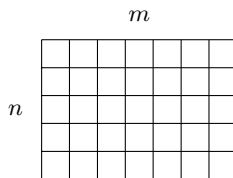


Fig. 1.3. The set Ω in Example 1.3

1.1 Partitions

If $(\alpha, \beta) \in \mathcal{C}_1$ then the number of γ which are specified associates of α and β are:

| | | | |
|-------------------------|------------------------|------------------------|------------------------|
| | $\mathcal{C}_1(\beta)$ | $\mathcal{C}_2(\beta)$ | $\mathcal{C}_3(\beta)$ |
| $\mathcal{C}_1(\alpha)$ | $m - 2$ | 0 | 0 |
| $\mathcal{C}_2(\alpha)$ | 0 | 0 | $n - 1$ |
| $\mathcal{C}_3(\alpha)$ | 0 | $n - 1$ | $(n - 1)(m - 2)$ |

The entries in the above table are the p_{ij}^1 .

If $(\alpha, \beta) \in \mathcal{C}_2$ then the number of γ which are specified associates of α and β are:

| | | | |
|-------------------------|------------------------|------------------------|------------------------|
| | $\mathcal{C}_1(\beta)$ | $\mathcal{C}_2(\beta)$ | $\mathcal{C}_3(\beta)$ |
| $\mathcal{C}_1(\alpha)$ | 0 | 0 | $m - 1$ |
| $\mathcal{C}_2(\alpha)$ | 0 | $n - 2$ | 0 |
| $\mathcal{C}_3(\alpha)$ | $m - 1$ | 0 | $(n - 2)(m - 1)$ |

The entries in the above table are the p_{ij}^2 .

Finally, if $(\alpha, \beta) \in \mathcal{C}_3$ then the number of γ which are specified associates of α and β are:

| | | | |
|-------------------------|------------------------|------------------------|------------------------|
| | $\mathcal{C}_1(\beta)$ | $\mathcal{C}_2(\beta)$ | $\mathcal{C}_3(\beta)$ |
| $\mathcal{C}_1(\alpha)$ | 0 | 1 | $m - 2$ |
| $\mathcal{C}_2(\alpha)$ | 1 | 0 | $n - 2$ |
| $\mathcal{C}_3(\alpha)$ | $m - 2$ | $n - 2$ | $(n - 2)(m - 2)$ |

and the entries in the above table are the p_{ij}^3 .

This is the *rectangular* association scheme $R(n, m)$ or $\underline{n} \times \underline{m}$. It has three associate classes. ■

Lemma 1.1 (i) $\sum_{i=0}^s a_i = |\Omega|;$

(ii) for every i and $k, \sum_j p_{ij}^k = a_i.$

Proof (i) The set Ω is the disjoint union of $\mathcal{C}_0(\alpha), \mathcal{C}_1(\alpha), \dots, \mathcal{C}_s(\alpha).$

(ii) Given any (α, β) in $\mathcal{C}_k,$ the set $\mathcal{C}_i(\alpha)$ is the disjoint union of the sets $\mathcal{C}_i(\alpha) \cap \mathcal{C}_j(\beta)$ for $j = 0, 1, \dots, s.$ ■

Thus it is sufficient to check constancy of the a_i for all but one value of $i,$ and, for each pair $(i, k),$ to check the constancy of p_{ij}^k for all but one value of $j.$

Thus construction of tables like those above is easier if we include a row for $\mathcal{C}_0(\alpha)$ and a column for $\mathcal{C}_0(\beta).$ Then the figures in the i -th row

and column must sum to a_i , so we begin by putting these totals in the margins of the table, then calculate the easier entries (remembering that the table must be symmetric), then finish off by subtraction.

Example 1.4 Let Ω consist of the vertices of the Petersen graph, which is shown in Figure 1.4. Let \mathcal{C}_1 consist of the edges of the graph and

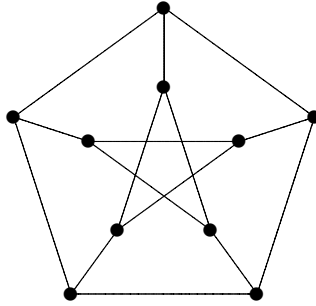


Fig. 1.4. The Petersen graph

\mathcal{C}_2 consist of the non-edges (that is, of those pairs of distinct vertices which are not joined by an edge). Inspection of the graph shows that every vertex is joined to three others and so $a_1 = 3$. It follows that $a_2 = 10 - 1 - 3 = 6$.

If $\{\alpha, \beta\}$ is an edge, we readily obtain the partial table

| | $\mathcal{C}_0(\beta)$ | $\mathcal{C}_1(\beta)$ | $\mathcal{C}_2(\beta)$ | |
|-------------------------|------------------------|------------------------|------------------------|----|
| $\mathcal{C}_0(\alpha)$ | 0 | 1 | 0 | 1 |
| $\mathcal{C}_1(\alpha)$ | 1 | 0 | | 3 |
| $\mathcal{C}_2(\alpha)$ | 0 | | | 6 |
| | 1 | 3 | 6 | 10 |

because there are no triangles in the graph. To obtain the correct row and column totals, this must be completed as

| | $\mathcal{C}_0(\beta)$ | $\mathcal{C}_1(\beta)$ | $\mathcal{C}_2(\beta)$ | |
|-------------------------|------------------------|------------------------|------------------------|----|
| $\mathcal{C}_0(\alpha)$ | 0 | 1 | 0 | 1 |
| $\mathcal{C}_1(\alpha)$ | 1 | 0 | 2 | 3 |
| $\mathcal{C}_2(\alpha)$ | 0 | 2 | 4 | 6 |
| | 1 | 3 | 6 | 10 |

We work similarly for the case that $\{\alpha, \beta\}$ is not an edge. If you are

familiar with the Petersen graph you will know that every pair of vertices which are not joined by an edge are both joined to exactly one vertex; if you are not familiar with the graph and its symmetries, you should check that this is true. Thus $p_{11}^2 = 1$. This gives the middle entry of the table, and the three entries in the bottom right-hand corner (p_{12}^2 , p_{21}^2 and p_{22}^2) can be calculated by subtraction.

| | | | | |
|-------------------------|------------------------|------------------------|------------------------|----|
| | $\mathcal{C}_0(\beta)$ | $\mathcal{C}_1(\beta)$ | $\mathcal{C}_2(\beta)$ | |
| $\mathcal{C}_0(\alpha)$ | 0 | 0 | 1 | 1 |
| $\mathcal{C}_1(\alpha)$ | 0 | 1 | 2 | 3 |
| $\mathcal{C}_2(\alpha)$ | 1 | 2 | 3 | 6 |
| | 1 | 3 | 6 | 10 |

So again we have an association scheme with two associate classes. ■

1.2 Graphs

Now that we have done the example with the Petersen graph, we need to examine graphs a little more formally. Recall that a finite graph is a finite set Γ , whose elements are called *vertices*, together with a set of 2-subsets of Γ called *edges*. (A ‘2-subset’ means a subset of size 2.) Strictly speaking, this is a *simple undirected* graph: in a non-simple graph, there may be two or more edges between a pair of vertices, while in a directed graph each edge is an ordered pair. Vertices γ and δ are said to be *joined by an edge* if $\{\gamma, \delta\}$ is an edge. The graph is *complete* if every 2-subset is an edge.

Example 1.4 suggests a second way of looking at association schemes. Imagine that all the edges in the Petersen graph are blue. For each pair of distinct vertices which are not joined, draw a red edge between them. We obtain a 2-colouring of the complete undirected graph K_{10} on ten vertices. Condition (iii) gives us a special property about the number of triangles of various types through an edge with a given colour.

Second definition of association scheme An *association scheme* with s associate classes on a finite set Ω is a colouring of the edges of the complete undirected graph with vertex-set Ω by s colours such that

(iii)' for all i, j, k in $\{1, \dots, s\}$ there is an integer p_{ij}^k such that, whenever $\{\alpha, \beta\}$ is an edge of colour k then

$$|\{\gamma \in \Omega : \{\alpha, \gamma\} \text{ has colour } i \text{ and } \{\gamma, \beta\} \text{ has colour } j\}| = p_{ij}^k;$$

(iv)' every colour is used at least once;

- (v)' there are integers a_i for i in $\{1, \dots, s\}$ such that each vertex is contained in exactly a_i edges of colour i .

The strange numbering of the conditions is to aid comparison with the previous definition. There is no need for an analogue of condition (i), because every edge consists of two distinct vertices, nor for an analogue of condition (ii), because we have specified that the graph be undirected. Condition (iii)' says that if we fix different vertices α and β , and colours i and j , then the number of triangles which consist of the edge $\{\alpha, \beta\}$ and an i -coloured edge through α and a j -coloured edge through β is exactly p_{ij}^k , where k is the colour of $\{\alpha, \beta\}$, irrespective of the choice of α and β (see Figure 1.5). We did not need a condition (iv) in the partition definition because we specified that the subsets in the partition be non-empty. Finally, because condition (iii)' does not deal with the analogue of the diagonal subset, we have to put in condition (v)' explicitly. If we assume that none of the colours used for the edges is white, we can colour all the vertices white (colour 0). Then we recover exactly the parameters of the first kind, with $a_0 = 1$, $p_{ii}^0 = a_i$, $p_{i0}^i = p_{0i}^i = 1$, and $p_{ij}^0 = p_{i0}^j = p_{0j}^i = 0$ if $i \neq j$.

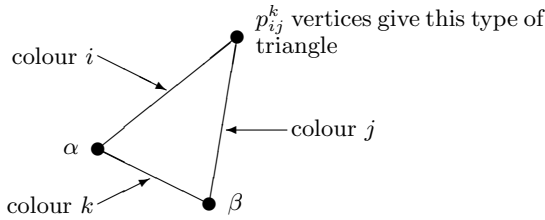


Fig. 1.5. Condition (iii)'

In fact, condition (v)' is redundant. For suppose that α and β are different vertices and let k be the colour of edge $\{\alpha, \beta\}$. For each i and j , there are exactly p_{ij}^k triangles consisting of $\{\alpha, \beta\}$, an i -coloured edge through α and a j -coloured edge through β . Of course, the edge $\{\alpha, \beta\}$ is an *unordered* pair, so there are also p_{ij}^k triangles consisting of $\{\alpha, \beta\}$, an i -coloured edge through β and a j -coloured edge through α : that is, $p_{ij}^k = p_{ji}^k$. If $k \neq i$ then the number of i -coloured edges through α and the number of i -coloured edges through β are both equal to $\sum_{j=1}^s p_{ij}^k$, as in Lemma 1.1(ii); if $k = i$ then we have to include the edge $\{\alpha, \beta\}$ itself so the number is $\sum_{j=1}^s p_{ij}^k + 1$.

Lemma 1.2 *If $|\Omega|$ is odd then all the non-trivial valencies are even.*

Proof There are a_i edges of colour i through each vertex. If $i \neq 0$ then each of these edges contains two vertices, and so there are $|\Omega| a_i/2$ such edges altogether. ■

Example 1.5 Let Ω be the set of the eight vertices of the cube. Colour the edges of the cube yellow, the main diagonals red and the face diagonals black. In Figure 1.6 the yellow edges are shown by solid lines, the red edges by dashed lines, and the black edges are omitted. If you find this picture hard to follow, take any convenient cuboid box, draw black lines along the face diagonals and colour the edges yellow.

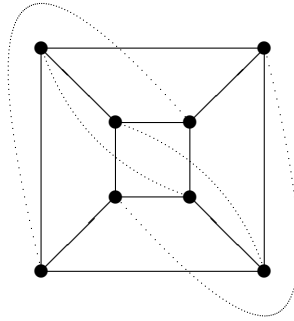


Fig. 1.6. The cube: solid lines are yellow, dashed ones are red

Every vertex is in three yellow edges, one red edge and three black ones, so $a_{\text{yellow}} = 3$, $a_{\text{red}} = 1$ and $a_{\text{black}} = 3$. The values p_{ij}^{yellow} are the entries in

$$\begin{array}{c}
 \text{white} \\
 \text{yellow} \\
 \text{red} \\
 \text{black}
 \end{array}
 \begin{bmatrix}
 \text{white} & \text{yellow} & \text{red} & \text{black} \\
 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & 2 \\
 0 & 0 & 0 & 1 \\
 0 & 2 & 1 & 0
 \end{bmatrix},$$

the values p_{ij}^{red} are the entries in

$$\begin{array}{c}
 \text{white} \\
 \text{yellow} \\
 \text{red} \\
 \text{black}
 \end{array}
 \begin{bmatrix}
 \text{white} & \text{yellow} & \text{red} & \text{black} \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 3 \\
 1 & 0 & 0 & 0 \\
 0 & 3 & 0 & 0
 \end{bmatrix}$$

and the values p_{ij}^{black} are the entries in

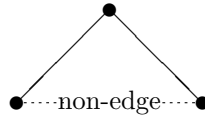
$$\begin{array}{l}
 \text{white} \\
 \text{yellow} \\
 \text{red} \\
 \text{black}
 \end{array}
 \begin{bmatrix}
 & \text{white} & \text{yellow} & \text{red} & \text{black} \\
 \left[\begin{array}{cccc}
 0 & 0 & 0 & 1 \\
 0 & 2 & 1 & 0 \\
 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & 2
 \end{array} \right]
 \end{bmatrix}
 .$$

Thus we have an association scheme with three associate classes. ■

If an association scheme has two associate classes, we can regard the two colours as ‘visible’ and ‘invisible’, as in Example 1.4. The graph formed by the visible edges is said to be *strongly regular*.

Definition A finite graph is *strongly regular* if

- (a) it is *regular* in the sense that every vertex is contained in the same number of edges;
- (b) every edge is contained in the same number of triangles;
- (c) every non-edge is contained in the same number of configurations like



- (d) it is neither complete (all pairs are edges) nor null (no pairs are edges).

1.3 Matrices

Given a field F , the set F^Ω of functions from Ω to F forms a vector space. Addition is defined pointwise: for f and g in F^Ω ,

$$(f + g)(\omega) = f(\omega) + g(\omega) \quad \text{for } \omega \text{ in } \Omega.$$

Scalar multiplication is also defined pointwise: for λ in F and f in F^Ω ,

$$(\lambda f)(\omega) = \lambda(f(\omega)) \quad \text{for } \omega \text{ in } \Omega.$$

For ω in Ω , let χ_ω be the characteristic function of ω ; that is

$$\begin{aligned}
 \chi_\omega(\omega) &= 1 \\
 \chi_\omega(\alpha) &= 0 \quad \text{for } \alpha \text{ in } \Omega \text{ with } \alpha \neq \omega.
 \end{aligned}$$