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Quantum fields

Quantum field theory is a necessary tool for the quantum mechanical description of processes that allow for transitions between states which differ in their particle content. Quantum field theory is thus quantum mechanics of an arbitrary number of particles. It is therefore mandatory for relativistic quantum theory since relativistic kinematics allows for creation and annihilation of particles in accordance with the formula for equivalence of energy and mass. Relativistic quantum theory is thus inherently dealing with many-body systems. One may, however, wonder why quantum field theoretic methods are so prevalent in condensed matter theory, which considers non-relativistic many-body systems. The reason is that, though not mandatory, it provides an efficient way of respecting the quantum statistics of the particles, i.e. the states of identical fermions or bosons must be antisymmetric and symmetric, respectively, under the interchange of pairs of identical particles. Furthermore, the treatment of spontaneously symmetry broken states, such as superfluids, is facilitated; not to mention critical phenomena in connection with phase transitions. Furthermore, the powerful functional methods of field theory, and methods such as the renormalization group, can by use of the non-equilibrium field theory technique be extended to treat non-equilibrium states and thereby transport phenomena.

It is useful to delve once into the underlying mathematical structure of quantum field theory, but the upshot of this chapter will be very simple: just as in quantum mechanics, where the transition operators, $|\phi\rangle\langle\psi|$, contain the whole content of quantum kinematics, and the *bra* and *ket* annihilate and create states in accordance with

$$(|\phi\rangle\langle\psi|)\,|\chi\rangle \quad = \quad \langle\psi|\chi\rangle\,|\phi\rangle \tag{1.1}$$

we shall find that in quantum field theory two types of operators do the same job. One of these operators, the creation operator, $a^\dagger$, is similar in nature to the *ket* in the transition operator, and the other, the annihilation operator, a, is similar to the action of the *bra* in Eq. (1.1), annihilating the state it operates on. Then the otherwise messy obedience of the quantum statistics of particles becomes a trivial matter expressed through the anti-commutation or commutation relations of the creation and annihilation operators.

1.1 Quantum mechanics

A short discussion of quantum mechanics is first given, setting the scene for the notation. In quantum mechanics, the state of a physical system is described by a vector, $|\psi\rangle$, providing a complete description of the system. The description is unique modulo a phase factor, i.e. the state of a physical system is properly represented by a ray, the equivalence class of vectors $e^{i\varphi}|\psi\rangle$, differing only by an overall phase factor of modulo one.

We consider first a single particle. Of particular intuitive importance are the states where the particle is definitely at a given spatial position, say $\mathbf{x}$, the corresponding state vector being denoted by $|\mathbf{x}\rangle$. The projection of an arbitrary state onto such a position state, the scalar product between the states,

$$\psi(\mathbf{x}) \;=\; \langle\mathbf{x}|\psi\rangle \,, \tag{1.2}$$

specifies the probability amplitude, the so-called wave function, whose absolute square is the probability for the event that the particle is located at the position in question.[1] The states of definite spatial positions are delta normalized

$$\langle\mathbf{x}|\mathbf{x}'\rangle \;=\; \delta(\mathbf{x}-\mathbf{x}') \,. \tag{1.3}$$

Of equal importance is the complementary representation in terms of the states of definite momentum, the corresponding state vectors denoted by $|\mathbf{p}\rangle$. Analogous to the position states they form a complete set or, equivalently, they provide a resolution of the identity operator, $\hat{I}$, in terms of the momentum state projection operators

$$\int d\mathbf{p}\; |\mathbf{p}\rangle\langle\mathbf{p}| \;=\; \hat{I} \,. \tag{1.4}$$

The appearance of an integral in Eq. (1.4) assumes space to be infinite, and the (conditional) probability amplitude for the event of the particle to be at position $\mathbf{x}$ *given* it has momentum $\mathbf{p}$ is specified by the plane wave function

$$\langle\mathbf{x}|\mathbf{p}\rangle \;=\; \frac{1}{(2\pi\hbar)^{3/2}}\, e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \,, \tag{1.5}$$

the transformation between the complementary representations being Fourier transformation. The states of definite momentum are therefore also delta normalized[2]

$$\langle\mathbf{p}|\mathbf{p}'\rangle \;=\; \delta(\mathbf{p}-\mathbf{p}') \,. \tag{1.6}$$

The possible physical momentum values are represented as eigenvalues, $\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$, of the operator

$$\hat{\mathbf{p}} \;=\; \int d\mathbf{p}\; \mathbf{p}\, |\mathbf{p}\rangle\langle\mathbf{p}| \tag{1.7}$$

[1] Treating space as a continuum, the relevant quantity is of course the probability for the particle being in a small volume around the position in question, $P(\mathbf{x})\Delta\mathbf{x} = |\psi(\mathbf{x})|^2\Delta\mathbf{x}$, the absolute square of the wave function denoting a probability *density*.

[2] If the particle is confined in space, say confined in a box as often assumed, the momentum states are Kronecker normalized, $\langle\mathbf{p}|\mathbf{p}'\rangle = \delta_{\mathbf{p},\mathbf{p}'}$.

representing the physical quantity *momentum*. Similarly for the position of a particle. The average value of a physical quantity is thus specified by the matrix element of its corresponding operator, say the average position in state $|\psi\rangle$ is given by the three real numbers composing the vector $\langle\psi|\hat{\mathbf{x}}|\psi\rangle$. In physics it is customary to interpret a scalar product as the value of the *bra*, a linear functional on the state vector space, on the vector, *ket*, in question.[3]

The complementarity of the position and momentum descriptions is also expressed by the commutator, $[\hat{\mathbf{x}}, \hat{\mathbf{p}}] \equiv \hat{\mathbf{x}}\,\hat{\mathbf{p}} - \hat{\mathbf{p}}\,\hat{\mathbf{x}}$, of the operators representing the two physical quantities, being the c-number specified by the quantum of action

$$[\hat{\mathbf{x}}, \hat{\mathbf{p}}] \;=\; i\hbar \,. \tag{1.8}$$

The fundamental position and momentum representations refer only to the kinematical structure of quantum mechanics. The dynamics of a system is determined by the Hamiltonian $\hat{H} = H(\hat{\mathbf{p}}, \hat{\mathbf{x}})$, the operator specified according to the correspondence principle by Hamilton's function $H(\hat{\mathbf{p}}, \hat{\mathbf{x}})$, i.e. for a non-relativistic particle of mass m in a potential $V(\mathbf{x})$ the Hamiltonian, the energy operator, is

$$\hat{H} \;=\; \frac{\hat{\mathbf{p}}^2}{2m} \;+\; V(\hat{\mathbf{x}}) \,. \tag{1.9}$$

It can often be convenient to employ the eigenstates of the Hamiltonian

$$\hat{H}\,|\epsilon_\lambda\rangle \;=\; \epsilon_\lambda\,|\epsilon_\lambda\rangle \,. \tag{1.10}$$

The completeness of the states of definite energy, $|\epsilon_\lambda\rangle$, is specified by *their* resolution of the identity

$$\sum_\lambda |\epsilon_\lambda\rangle\langle\epsilon_\lambda| = \hat{I} \tag{1.11}$$

here using a notation corresponding to the case of a discrete spectrum.

At each instant of time a complete description is provided by a state vector, $|\psi(t)\rangle$, thereby defining an operator, the time-evolution operator connecting state vectors at different times

$$|\psi(t)\rangle \;=\; \hat{U}(t,t')\,|\psi(t')\rangle \,. \tag{1.12}$$

Conservation of probability, conservation of the length of a state vector, or its normalized scalar product $\langle\psi(t)|\psi(t)\rangle = 1$, under time evolution, determines the evolution operator to be unitary, $U^{-1}(t,t') = U^\dagger(t,t')$. The dynamics is given by the Schrödinger equation

$$i\hbar\,\frac{d|\psi(t)\rangle}{dt} \;=\; \hat{H}\,|\psi(t)\rangle \tag{1.13}$$

and for an isolated system the evolution operator is thus the unitary operator

$$\hat{U}(t,t') \;=\; e^{-\frac{i}{\hbar}\hat{H}(t-t')} \,. \tag{1.14}$$

Here we have presented the operator calculus approach to quantum dynamics, the equivalent path integral approach is presented in Appendix A.

[3]For a detailed introduction to quantum mechanics we direct the reader to chapter 1 in reference [1].

In order to describe a physical problem we need to specify particulars, typically in the form of an initial condition. Such general initial condition problems can be solved through the introduction of the Green's function. The Green's function $G(\mathbf{x}, t; \mathbf{x}', t')$ represents the solution to the Schrödinger equation for the particular initial condition where the particle is definitely at position $\mathbf{x}'$ at time t'

$$\lim_{t \searrow t'} \psi(\mathbf{x}, t) \;=\; \delta(\mathbf{x} - \mathbf{x}') \;=\; \langle \mathbf{x}, t' | \mathbf{x}', t' \rangle \; . \tag{1.15}$$

The solution of the Schrödinger equation corresponding to this initial condition therefore depends parametrically on $\mathbf{x}'$ (and t'), and is by definition the conditional probability density amplitude for the dynamics in question[4]

$$\psi_{\mathbf{x}',t'}(\mathbf{x}, t) \;=\; \langle \mathbf{x}, t | \mathbf{x}', t' \rangle \;=\; \langle \mathbf{x} | \hat{U}(t, t') | \mathbf{x}' \rangle \;\equiv\; G(\mathbf{x}, t; \mathbf{x}', t') \; . \tag{1.16}$$

The Green's function, defined to be a solution of the Schrödinger equation, satisfies

$$\left(i\hbar \frac{\partial}{\partial t} \;-\; H(-i\hbar\nabla_{\mathbf{x}}, \mathbf{x}) \right) G(\mathbf{x}, t; \mathbf{x}', t') = 0 \tag{1.17}$$

where, according to Eq. (1.3), the Hamiltonian in the position representation, H, is specified by the position matrix elements of the Hamiltonian

$$\langle \mathbf{x} | \hat{H} | \mathbf{x}' \rangle \;=\; H(-i\hbar\nabla_{\mathbf{x}}, \mathbf{x}) \, \delta(\mathbf{x} - \mathbf{x}') \; . \tag{1.18}$$

The Green's function, G, is the kernel of the Schrödinger equation on integral form (being a first order differential equation in time)

$$\psi(\mathbf{x}, t) \;=\; \int \! d\mathbf{x}' \, G(\mathbf{x}, t; \mathbf{x}', t') \, \psi(\mathbf{x}', t') \tag{1.19}$$

as identified in terms of the matrix elements of the evolution operator by using the resolution of the identity in terms of the position basis states

$$\langle \mathbf{x} | \psi(t) \rangle \;=\; \int \! d\mathbf{x}' \langle \mathbf{x} | \hat{U}(t, t') | \mathbf{x}' \rangle \langle \mathbf{x}' | \psi(t') \rangle \; . \tag{1.20}$$

The Green's function propagates the wave function, and we shall therefore also refer to the Green's function as the propagator. It completely specifies the quantum dynamics of the particle.

We note that the partition function of thermodynamics and the trace of the evolution operator are related by analytical continuation:

$$\begin{aligned} Z \;&=\; \mathrm{Tr}\, e^{-\hat{H}/kT} = \int \! d\mathbf{x} \, \langle \mathbf{x} | e^{-\hat{H}/kT} | \mathbf{x} \rangle \;=\; \mathrm{Tr}\, \hat{U}(-i\hbar/kT, 0) \\ &=\; \int \! d\mathbf{x} \, G(\mathbf{x}, -i\hbar/kT; \mathbf{x}, 0) \end{aligned} \tag{1.21}$$

[4]In the continuum limit the Green's function is not a normalizable solution of the Schrödinger equation, as is clear from Eq. (1.15).

showing that the partition function is obtained from the propagator at the imaginary time $\tau = -i\hbar/kT$. The formalisms of thermodynamics, i.e. equilibrium statistical mechanics, and quantum mechanics are thus equivalent, a fact we shall take advantage of throughout. The physical significance is the formal equivalence of quantum and thermal fluctuations.

Quantum mechanics can be formulated without the use of operators, viz. using Feynman's path integral formulation. In Appendix A, the path integral expressions for the propagator and partition function for a single particle are obtained. Various types of Green's functions and their properties for the case of a single particle are discussed in Appendix C, and their analytical properties are considered in Appendix D.

1.2 *N*-particle system

Next we consider a physical system consisting of N particles. If the particles in an assembly are distinguishable, i.e. different species of particles, an orthonormal basis in the N-particle state space $H^{(N)} = H_1 \otimes H_2 \otimes \cdots \otimes H_N$ is the (tensor) product states, for example specified in terms of the momentum quantum numbers of the particles

$$|\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\rangle \equiv |\mathbf{p}_1\rangle \otimes |\mathbf{p}_2\rangle \otimes \cdots \otimes |\mathbf{p}_N\rangle \equiv |\mathbf{p}_1\rangle|\mathbf{p}_2\rangle \cdots |\mathbf{p}_N\rangle \,. \tag{1.22}$$

We follow the custom of suppressing the tensorial notation.

Formally everything in the following, where an N-particle system is considered, is equivalent no matter which complete set of single-particle states are used. In practice the choice follows from the context, and to be specific we shall mainly explicitly employ the momentum states, the choice convenient in practice for a spatially translational invariant system.[5] These states are eigenstates of the momentum operators

$$\hat{\mathbf{p}}_i \, |\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\rangle = \mathbf{p}_i \, |\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\rangle \,, \tag{1.23}$$

where tensorial notation for operators are suppressed, i.e.

$$\hat{\mathbf{p}}_i = \hat{I}_1 \otimes \cdots \hat{I}_{i-1} \otimes \hat{\mathbf{p}}_i \otimes \hat{I}_{i+1} \otimes \cdots \hat{I}_N \,, \tag{1.24}$$

each operating in the one-particle subspace dictated by its index. In particular the N-particle momentum states are eigenstates of the total momentum operator

$$\hat{\mathbf{P}}_N = \sum_{i=1}^{N} \hat{\mathbf{p}}_i \tag{1.25}$$

[5] In the next sections we shall mainly use the momentum basis, and refer in the following to the quantum numbers labeling the one-particle states as *momentum*, although any complete set of quantum numbers could equally well be used. The N-tuple $(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ is a complete description of the N-particle system if the particles do not posses internal degrees of freedom. In the following, where we for example have electrons in mind, we suppress for simplicity of notation the spin labeling and simply assume it is absorbed in the momentum labeling. If the particles have additional internal degrees of freedom, such as color and flavor, these are included in a similar fashion. If more than one type of species is to be considered simultaneously the species type, say quark and gluon, must also be indicated.

corresponding to the total momentum eigenvalue

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i \, . \tag{1.26}$$

The position representation of the momentum states is specified by the plane wave functions, Eq. (1.5), the scalar product of the momentum states and the analogous N-particle states of definite positions being

$$\begin{aligned} \psi_{\mathbf{p}_1,\ldots,\mathbf{p}_N}(\mathbf{x}_1,\ldots,\mathbf{x}_N) &= \langle \mathbf{x}_1, \mathbf{x}_2, \ \ldots \ , \mathbf{x}_N | \mathbf{p}_1, \mathbf{p}_2, \ \ldots \ , \mathbf{p}_N \rangle = \prod_{i=1}^{N} \langle \mathbf{x}_i | \mathbf{p}_i \rangle \\ &= \left(\frac{1}{(2\pi\hbar)^{3/2}} \right)^N e^{\frac{i}{\hbar}\mathbf{p}_1\cdot\mathbf{x}_1} \, e^{\frac{i}{\hbar}\mathbf{p}_2\cdot\mathbf{x}_2} \ \ldots \ e^{\frac{i}{\hbar}\mathbf{p}_N\cdot\mathbf{x}_N} \, . \end{aligned} \tag{1.27}$$

1.2.1 Identical particles

For an assembly of identical particles a profound change in the above description is needed. In quantum mechanics true identity between objects are realized, viz. elementary particle species, say electrons, are profoundly identical, i.e. there exists nothing in Nature which can distinguish any two electrons. Identical particles are indistinguishable. States which differ only by two identical particles being interchanged are thus described by the same ray.[6] As a consequence of their indistinguishability, assemblies of identical particles are described by states which with respect to interchange of pairs of identical particles are either antisymmetric or symmetric

$$|\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\rangle \ = \ \pm \, |\mathbf{p}_2, \mathbf{p}_1, \ldots, \mathbf{p}_N\rangle \, , \tag{1.28}$$

this leaving the probability for a set of momenta of the particles, $P(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$, a function symmetric with respect to interchange of any pair of the identical particles.

A word on notation: the particle we call the *first* particle is in the momentum state specified by the first argument, and the particle we call the Nth particle is in the momentum state specified by the Nth argument. Particles whose states are symmetric with respect to interchange are called bosons , and for the antisymmetric case called fermions.[7]

[6] The quantum state with all of the electrons in the Universe interchanged will thus be the same as the present one. A radical invariance property of systems of identical particles!

[7] Quantum statistics and the spin degree of freedom of a particle are intimately connected as relativistic quantum field theory demands that bosons have integer spin, whereas particles with half-integer spin are fermions. This so-called spin-statistics connection seems in the present non-relativistic quantum theory quite mysterious, i.e. unintelligible. It only gets its explanation in the relativistic quantum theory, which we usually connect with high energy phenomena, where for any particle relativity, through Lorentz invariance, requires the existence of an anti-particle of the same mass and opposite charge (some neutral particles, such as the photon, are their own anti-particle). Then, in fermion anti-fermion pair production the particles must be antisymmetric with already existing particles as unitarity, i.e. conservation of probability, requires such a minus sign [2]. Historically, the exclusion principle, which is a direct consequence of Fermi statistics, was discovered by Pauli before the advent of relativistic quantum theory as a vehicle to explain the periodic properties of the elements. Pauli was also the first to show that the spin-statistics relation is a consequence of Lorentz invariance, causality and energy and norm positivity.

Any N-particle state $|\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\rangle$ can be mapped into a state which is either symmetric or antisymmetric with respect to interchange of any two particles. To obtain the symmetric state we simply apply the symmetrization operator $\hat{\mathcal{S}}$ which symmetrizes an N-particle state according to

$$\hat{\mathcal{S}}\,|\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\rangle \;=\; \frac{1}{N!}\sum_P |\mathbf{p}_{P_1}\rangle|\mathbf{p}_{P_2}\rangle \cdots |\mathbf{p}_{P_N}\rangle \tag{1.29}$$

and the antisymmetrization operator $\hat{\mathcal{A}}$ antisymmetrizes according to

$$\hat{\mathcal{A}}\,|\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\rangle \;=\; \frac{1}{N!}\sum_P (-1)^{\zeta_P}|\mathbf{p}_{P_1}\rangle|\mathbf{p}_{P_2}\rangle \cdots |\mathbf{p}_{P_N}\rangle\,. \tag{1.30}$$

The summations are over all permutations P of the particles. Permutations form a group, and any permutation can be build by successive transpositions which only permute a pair. In the case of antisymmetrization, each term appears with the sign of the permutation in question

$$\mathrm{sign}(P) \;=\; \prod_{1\leq i<j\leq N} \frac{j-i}{P_j - P_i}\,. \tag{1.31}$$

We have written this in terms of the number ζ_P which counts the number of transpositions needed to build the permutation P, since $\mathrm{sign}(P) = (-1)^{\zeta_P}$.

If the single-particle state labels in the N-particle state to be symmetrized on the left in Eq. (1.29) are permuted, the same symmetrized state results, since if P' can be any of the $N!$ permutations, then $P'P$ for fixed permutation P will run through them all, $\hat{\mathcal{S}}\,|\mathbf{p}_{P_1}, \mathbf{p}_{P_2}, \ldots\, \mathbf{p}_{P_N}\rangle = \hat{\mathcal{S}}\,|\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\rangle$.

We note that the sign of a product of permutations, $Q = P'P$, equals the product of the signs of the two permutations, $\mathrm{sign}(Q) = \mathrm{sign}(P')\cdot\mathrm{sign}(P)$, and a permutation and its inverse have the same sign (owing to their equal number of transpositions), $\zeta_{P^{-1}} = \zeta_P$. Antisymmetrization of a permuted state gives the same antisymmetric state multiplied by the sign of the permutation permuting the original N-particle state since

$$\hat{\mathcal{A}}\,|\mathbf{p}_{P_1}, \mathbf{p}_{P_2}, \ldots\, \mathbf{p}_{P_N}\rangle \;=\; \frac{1}{N!}\sum_{P'} (-1)^{\zeta_{P'}}|\mathbf{p}_{Q_1}\rangle|\mathbf{p}_{Q_2}\rangle \cdots |\mathbf{p}_{Q_N}\rangle \tag{1.32}$$

and as P' runs through all the permutations so does $Q = P'P$, and therefore

$$\begin{aligned}\hat{\mathcal{A}}\,|\mathbf{p}_{P_1}, \mathbf{p}_{P_2}, \ldots\, \mathbf{p}_{P_N}\rangle \;&=\; (-1)^{\zeta_P}\frac{1}{N!}\sum_{Q} (-1)^{\zeta_Q}|\mathbf{p}_{Q_1}\rangle|\mathbf{p}_{Q_2}\rangle \cdots |\mathbf{p}_{Q_N}\rangle \\ &=\; (-1)^{\zeta_P}\,\hat{\mathcal{A}}\,|\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\rangle\,.\end{aligned} \tag{1.33}$$

Therefore, if any two single-particle states are identical, the antisymmetrized state vector equals the zero vector, since the two states obtained by permuting the two identical labels are identical and yet upon antisymmatrization they differ by a minus sign, i.e. Pauli's exclusion principle for fermions: no two fermions can occupy the same state.

Further, according to Eq. (1.33), applying the antisymmetrization operator twice

$$\begin{aligned}\hat{\mathcal{A}}^2\,|\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_N\rangle &= \hat{\mathcal{A}}\frac{1}{N!}\sum_P(-1)^{\zeta_P}|\mathbf{p}_{P_1}\rangle|\mathbf{p}_{P_2}\rangle\cdots|\mathbf{p}_{P_N}\rangle\\ &= \frac{1}{N!}\sum_P(-1)^{\zeta_P}(-1)^{\zeta_P}\hat{\mathcal{A}}\,|\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_N\rangle\\ &= \hat{\mathcal{A}}\,|\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_N\rangle \qquad (1.34)\end{aligned}$$

gives the same state as applying it only once, i.e. the symmetrization operators are projectors, $\hat{\mathcal{A}}^2=\hat{\mathcal{A}}$, $\hat{\mathcal{S}}^2=\hat{\mathcal{S}}$. The presence of the factor $1/N!$ in the definitions, Eq. (1.29) and Eq. (1.30), is thus there to ensure the operators are normalized projectors. Representing mutually exclusive symmetry properties, they are orthogonal projectors, their product is the operator that maps any vector onto the zero vector

$$\hat{\mathcal{A}}\,\hat{\mathcal{S}}=\hat{0}=\hat{\mathcal{S}}\,\hat{\mathcal{A}} \qquad (1.35)$$

since symmetrizing an antisymmetric state, or vice versa, gives the zero vector.

The symmetrization operators are hermitian, $\hat{\mathcal{A}}^\dagger=\hat{\mathcal{A}}$, $\hat{\mathcal{S}}^\dagger=\hat{\mathcal{S}}$, as verified for example for $\hat{\mathcal{A}}$ by first noting that according to the definition of the adjoint operator

$$\begin{aligned}\langle\mathbf{p}_1,\,\ldots\,,\mathbf{p}_N|\hat{\mathcal{A}}^\dagger|\mathbf{p}'_1,\mathbf{p}'_2,\,\ldots\,,\mathbf{p}'_N\rangle &= \langle\mathbf{p}'_1,\,\ldots\,,\mathbf{p}'_N|\hat{\mathcal{A}}|\mathbf{p}_1,\mathbf{p}_2,\,\ldots\,,\mathbf{p}_N\rangle^*\\ &= \frac{1}{N!}\sum_P(-1)^{\zeta_P}\langle\mathbf{p}'_1|\mathbf{p}_{P_1}\rangle^*\,\cdots\,\langle\mathbf{p}'_N|\mathbf{p}_{P_N}\rangle^*\\ &= \frac{(-1)^{\zeta_S}}{N!}\langle\mathbf{p}_{S_1}|\mathbf{p}'_1\rangle\,\cdots\,\langle\mathbf{p}_{S_N}|\mathbf{p}'_N\rangle \qquad (1.36)\end{aligned}$$

the matrix element being nonzero only if the set $\{\mathbf{p}'_i\}_{i=1,\ldots,N}$ is a permutation of the set $\{\mathbf{p}_i\}_{i=1,\ldots,N}$, S being the permutation that brings the set $\{\mathbf{p}_i\}_{i=1,\ldots,N}$ into the set $\{\mathbf{p}'_i\}_{i=1,\ldots,N}$, $\mathbf{p}_{S_i}=\mathbf{p}'_i$. Permuting both sets of indices by the inverse permutation S^{-1} of S, and using that $\zeta_{S^{-1}}=\zeta_S$, we get

$$\begin{aligned}\langle\mathbf{p}_1,\,\ldots\,,\mathbf{p}_N|\hat{\mathcal{A}}^\dagger|\mathbf{p}'_1,\mathbf{p}'_2,\,\ldots\,,\mathbf{p}'_N\rangle &= \frac{1}{N!}(-1)^{\zeta_{S^{-1}}}\langle\mathbf{p}_1|\mathbf{p}'_{S^{-1}_1}\rangle\,\cdots\,\langle\mathbf{p}_N|\mathbf{p}'_{S^{-1}_N}\rangle\\ &= \frac{1}{N!}\sum_P(-1)^{\zeta_P}\langle\mathbf{p}_1,\,\ldots\,,\mathbf{p}_N|\mathbf{p}'_{P_1},\,\ldots\,,\mathbf{p}'_{P_N}\rangle\\ &= \langle\mathbf{p}_1,\,\ldots\,,\mathbf{p}_N|\hat{\mathcal{A}}|\mathbf{p}'_1,\,\ldots\,,\mathbf{p}'_N\rangle\,. \qquad (1.37)\end{aligned}$$

Exercise 1.1. Show that the adjoint of a product of linear operators A and B is the product of their adjoint operators in opposite sequence

$$(A\,B)^\dagger = B^\dagger\,A^\dagger \qquad (1.38)$$

and generalize to the case of an arbitrary number of operators.

Exercise 1.2. The vector space of state vectors, the *kets*, and the dual space of linear functionals on the state space, the *bras*, are isomorphic vector spaces, which we express by the adjoint operation, $|\psi\rangle^\dagger = \langle\psi|$ and $\langle\psi|^\dagger = |\psi\rangle$. This mapping is anti-linear and isomorphic, and we use the same symbol as for the adjoint of an operator.

Show that for arbitrary state vectors and operators on the state space the relationship $(\hat{X}|\psi\rangle)^\dagger = \langle\psi|\hat{X}^\dagger$. An operator being its own adjoint, $\hat{X}^\dagger = \hat{X}$, is said to be a hermitian operator and its eigenvalues are real, such operators being of primary importance in quantum mechanics.

Exercise 1.3. Show that the symmetrization operator, $\hat{\mathcal{S}}$, is hermitian.

The linear operators $\hat{\mathcal{S}}$ and $\hat{\mathcal{A}}$ project any state onto either of the two orthogonal subspaces of symmetric or antisymmetric states.[8] The state space for a physical system consisting of N identical particles is thus not $H^{(N)}$, the N-fold product of the one-particle state space, but either the symmetric subspace, $\mathcal{B}^{(N)}$, for bosons, or antisymmetric subspace, $\mathcal{F}^{(N)}$, for fermions, obtained by projecting the states of $H^{(N)}$ by either type of symmetrization operator depending on the statistics of the particles in question.

1.2.2 Kinematics of fermions

Let us introduce the orthogonal, normalized up to a phase factor, antisymmetric basis states in the antisymmetric N-particle state space $\mathcal{F}^{(N)}$

$$\begin{aligned} |\mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \cdots \wedge \mathbf{p}_N\rangle &\equiv \sqrt{N!}\,\hat{\mathcal{A}}\,|\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\rangle \\ &= \frac{1}{\sqrt{N!}} \sum_P (-1)^{\zeta_P} |\mathbf{p}_{P_1}\rangle \otimes |\mathbf{p}_{P_2}\rangle \otimes \cdots \otimes |\mathbf{p}_{P_N}\rangle \\ &= \frac{1}{\sqrt{N!}} \sum_P (-1)^{\zeta_P} |\mathbf{p}_{P_1}\rangle |\mathbf{p}_{P_2}\rangle \cdots |\mathbf{p}_{P_N}\rangle \\ &= \frac{1}{\sqrt{N!}} \sum_P (-1)^{\zeta_P} |\mathbf{p}_{P_1}, \mathbf{p}_{P_2}, \ldots, \mathbf{p}_{P_N}\rangle\,. \end{aligned} \tag{1.39}$$

We demonstrate that they are orthogonal by using the properties of the antisymmetrization operator (we first for simplicity of the Kronecker function assume box normalization, i.e, the momentum values are discrete)

$$\begin{aligned} \langle \mathbf{p}_1 \wedge \cdots \wedge \mathbf{p}_N | \mathbf{p}'_1 \wedge \cdots \wedge \mathbf{p}'_N\rangle &= N!\langle \mathbf{p}_1, \ldots, \mathbf{p}_N|\hat{\mathcal{A}}^\dagger \hat{\mathcal{A}}|\mathbf{p}'_1, \ldots, \mathbf{p}'_N\rangle \\ &= N!\langle \mathbf{p}_1, \ldots, \mathbf{p}_N|\hat{\mathcal{A}}|\mathbf{p}'_1, \ldots, \mathbf{p}'_N\rangle \end{aligned}$$

[8] Only for the case of *two* particles do the two subspaces of symmetric and antisymmetric states span the original state space, $H^{(2)} = H \otimes H$. In general, the other subspaces for the case of more than two particles do not seem to be state spaces for systems of identical particles.

$$= \langle \mathbf{p}_1, \ldots, \mathbf{p}_N | \sum_P (-1)^{\zeta_P} |\mathbf{p}'_{P_1}, \ldots, \mathbf{p}'_{P_N}\rangle$$

$$= \begin{cases} (-1)^{\zeta_S} & \{\mathbf{p}'\}_i \equiv \{\mathbf{p}\}_i \\ 0 & \text{otherwise} \end{cases} \tag{1.40}$$

where $\{\mathbf{p}'_i\}_{i=1,\ldots,N} \equiv \{\mathbf{p}_i\}_{i=1,\ldots,N}$ is short for *the labels* $\{\mathbf{p}'_i\}_{i=1,\ldots,N}$ *being a permutation of the labels* $\{\mathbf{p}_i\}_{i=1,\ldots,N}$, and S the permutation that takes the set $\{\mathbf{p}_i\}_{i=1,\ldots,N}$ into $\{\mathbf{p}'_i\}_{i=1,\ldots,N}$, $\mathbf{p}_{S_i} = \mathbf{p}'_i$. Or simply in words, only if the primed set of momenta is a permutation of the unprimed set is the scalar product of the states nonzero (we have of course assumed that all momenta are different since otherwise for fermions the vector is the zero-vector).

Incidentally we have

$$\langle \mathbf{p}_1 \wedge \mathbf{p}_2, \wedge \cdots \wedge \mathbf{p}_N | \, \mathbf{p}'_1, \mathbf{p}'_2, .., \mathbf{p}'_N\rangle = \begin{cases} \frac{1}{\sqrt{N!}} (-1)^{\zeta_S} & \{\mathbf{p}'\}_i \equiv \{\mathbf{p}\}_i \\ 0 & \text{otherwise} \end{cases} \tag{1.41}$$

expressing that additional permutations are redundant, for example an additional antisymmetrization is redundant as expressed by the second equality sign in Eq. (1.40), or equivalently that the symmetrization operators are hermitian projectors.

The scalar product of antisymmetric states is the determinant of the $N \times N$ matrix with entries $\langle \mathbf{p}_i | \mathbf{p}'_j\rangle$

$$\langle \mathbf{p}_1 \wedge \cdots \wedge \mathbf{p}_N | \mathbf{p}'_1 \wedge \cdots \wedge \mathbf{p}'_N\rangle = \det(\langle \mathbf{p}_i | \mathbf{p}'_j\rangle)$$

$$= \sum_P (-1)^{\zeta_P} \, \langle \mathbf{p}_1 | \mathbf{p}'_{P_1}\rangle \cdots \langle \mathbf{p}_N | \mathbf{p}'_{P_N}\rangle \,, \tag{1.42}$$

the Slater determinant.

In the operator calculus perturbation theory for a single particle, the resolution of the identity plays a crucial efficient role. For an assembly of identical particles this role will be taken over by the commutation rules for the quantum fields we shall shortly introduce. The resolutions of the identity on the symmetrized subspaces reflect the redundancy of antisymmetrized or symmetrized states. Though not of much practical use, we include them for completeness (the resolution of the identity makes a short appearance in Section 3.1.1). The resolution of the identity on the antisymmetric state space can be written in terms of the N-state identity operator since the identity operator commutes with any operator

$$1 = \hat{\mathcal{A}} \, \hat{I}^{(N)} \, \hat{\mathcal{A}} = \hat{\mathcal{A}} \, \left(\hat{I}_1 \otimes \hat{I}_2 \otimes \hat{I}_3 \otimes \cdots \otimes \hat{I}_N\right) \, \hat{\mathcal{A}}^\dagger$$

$$= \hat{\mathcal{A}} \sum_{\mathbf{p}_1,\ldots,\mathbf{p}_N} |\mathbf{p}_1\rangle\langle\mathbf{p}_1| \otimes |\mathbf{p}_2\rangle\langle\mathbf{p}_2| \otimes \cdots \otimes |\mathbf{p}_N\rangle\langle\mathbf{p}_N| \, \hat{\mathcal{A}}^\dagger$$

$$= \hat{\mathcal{A}} \sum_{\mathbf{p}_1,\ldots,\mathbf{p}_N} |\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\rangle\langle\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N| \, \hat{\mathcal{A}}^\dagger$$