THE AUTONOMY OF MATHEMATICAL KNOWLEDGE

Most scholars think of David Hilbert's program as the most demanding and ideologically motivated attempt to provide a foundation for mathematics, and, because they see technical obstacles in the way of realizing the program's goals, they regard it as a failure. Against this view, Curtis Franks argues that Hilbert's deepest and most central insight was that mathematical techniques and practices do not need grounding in any philosophical principles. He weaves together an original historical account, philosophical analysis, and his own development of the meta-mathematics of weak systems of arithmetic to show that the true philosophical significance of Hilbert's program is that it makes the autonomy of mathematics evident. The result is a vision of the early history of modern logic that highlights the rich interaction between its conceptual problems and its technical development.

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Hilbert's Program Revisited

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For Lydia

The one and what I said about it make two,
and two and the original one make three.
If we go on in this way,
then even the cleverest mathematician can't tell
where we'll end,
much less an ordinary man.

If by moving from nonbeing to being we get to three,
how far will we get if we move from being to being?
Better not to move, but to let things be!

Chuang Tzu, *The Inner Chapters*

Contents

Preface *page* ix
Acknowledgments xii

1 A new science 1
- 1.1 Recovering Hilbert's thought 1
- 1.2 Freedom from nature 9
- 1.3 Freedom from philosophy 14
- 1.4 The wrong conclusion 25

2 David Hilbert's naturalism 29
- 2.1 Introduction 29
- 2.2 Anti-foundationalism 32
- 2.3 Mathematical autonomy 41
- 2.4 Formalism and finitism 47
- 2.5 Conclusion 59

3 Arithmetization 61
- 3.1 Introduction 61
- 3.2 Gödel's work as a contribution to Hilbert's program 64
- 3.3 The *Grundlagen der Geometrie* 68
- 3.4 Hilbert's proto-proof theory 73
- 3.5 Herbrand's reception of Hilbert 83
- 3.6 Investigating metatheory with arithmetic 89
- 3.7 Progress towards purity 98

4 Intensionality 105
- 4.1 Introduction 105
- 4.2 Gödel's second theorem and Hilbert's program 112
- 4.3 Feferman's approach 119
- 4.4 The inside/outside distinction 123
- 4.5 A theory-dependent interpretation of consistency 127
- 4.6 Conclusion 136

5 Interpreting Gödel's second incompleteness theorem for Q 139
- 5.1 Historical background 139
- 5.2 Pudlák's argument 142
- 5.3 Two routes to intensionality 145

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Curtis Franks
Frontmatter
More information

5.4 Ambiguity in interpretation 152
5.5 Concluding remark 167

6 Autonomy in context 169
6.1 The naturalistic turn 169
6.2 Wittgenstein's critique of the second-order 174
6.3 First-order meta-mathematics 184
6.4 Evidence of autonomy 191

References 200
Index 209

Cambridge University Press
978-0-521-18389-5 - The Autonomy of Mathematical Knowledge: Hilbert's Program Revisited
Curtis Franks
Frontmatter
More information

Preface

When I decided to bundle my recent historical, philosophical, and logical research together into a book, I considered two different approaches. One approach was to try to represent David Hilbert's foundational program exhaustively, to let my own findings simply shape the re-telling of a fairly familiar story. Another approach – the one that I ultimately preferred – was to center the book around what I think are both the most important and most overlooked aspects of Hilbert's program. There is no lack of high quality writing about Hilbert, nor of high quality development of his scientific innovations, but there seems to me to be a glaring oversight of one truly unique aspect of Hilbert's thought. So this book is a modest attempt to isolate, explain, and develop a single strain of Hilbert's philosophy. If the book inspires new interest and appreciation of Hilbert, then it has served its purpose.

The core of the book grew out of my doctoral research at the University of California's Department of Logic and Philosophy of Science in Irvine. The title of my doctoral thesis, "Mathematics speaks for itself," was a *double entendre*. It was meant to suggest two distinct but related themes. The first theme is that questions about mathematics that arise in philosophical reflection – questions about how and why its methods work – might be best addressed mathematically. I believe that this is so, and I claim that David Hilbert held the same view. In Chapter 2, I explain that Hilbert's program was primarily an effort to demonstrate that mathematics could answer questions about how its own methods work. Hilbert thought that if he could succeed at this, then he would have carved out a privileged position for mathematics among the sciences. Unlike other ways of knowing, the validity of ordinary mathematical methods would be seen to be independent of any philosophical theories of knowledge, as autonomous.

The second theme arises out of the first. Once one sees mathematics potentially providing its own foundations, one faces questions about the available ways for it to do so. The two most poignant issues are how a formal theory should refer to itself and how properties *about* a theory should be represented within that theory. I claim that standard answers to these questions are not sufficiently free from extra-mathematical, philosophical assumptions to speak fully to Hilbert's vision. In Chapter 3, I examine Hilbert's own attempt to settle these questions and explain why his attempt failed by his own standards. Then I turn to Jacques Herbrand's contribution to Hilbert's program and discuss the partial progress he made to Hilbert's goal of mathematical autonomy. Herbrand's work is known primarily for its purely mathematical accomplishments. I do not know of any detailed study on his philosophical perspective. However, a close look at his methods and remarks about the significance of his results reveals that he had a rich philosophical perspective, close in spirit to Hilbert's. In fact he viewed his own [1930a] Fundamental Theorem as a contribution to Hilbert's project of formulating questions of metatheory purely mathematically, and he even recognized the need for additional techniques of arithmetization (of the sort Gödel would later supply) to complement his own.

In Chapter 4, I rephrase the discussion in terms due to Solomon Feferman. It turns out that his notion of *intensionality* is precisely what a mathematical study of mathematics – in the spirit of Hilbert's original vision – requires. I examine the extent to which Gödel's and Herbrand's techniques of arithmetization are intensionally correct and suggest that a certain combination of the two works much better than either one on its own. Specifically, a Gödel-style encoding of the formulations of provability and consistency that result from Herbrand's theorem returns formulas that can be relativized to the computational strength of any arithmetical system. As a result one is able to pose questions about a system's metatheory to that system always in a way such that the system can understand the questions.

As an illustration of the applicability of these techniques, in Chapter 5 I apply the point of view from previous chapters to a specific problem in the philosophy of mathematics: whether the fact that a version of Gödel's second incompleteness theorem for Robinson's arithmetical theory Q can be understood as showing that Q cannot prove its own consistency. It is worth mentioning the chapter's focus on weak mathematical systems. I have often heard philosophers bemoan the attention that mathematical logic research, especially in recent years, gives to weak theories. My focus is on bounded and induction-free fragments of arithmetic, which are weak systems by any standard. Thus I must explain myself. I am in full sympathy with philosophers who see foundational studies as missing an essential point when focused solely on such weak systems. Weak fragments of arithmetic are not the theories that mathematicians ordinarily use, so it seems at first odd to suggest that a study of these theories can turn up "a foundation of mathematics." A singular focus on weak arithmetics appears to most philosophers just as the obscure preoccupations of his day appeared to Laurence Sterne, as "common-place infirmity of the greatest mathematicians! working with might and main at the demonstration, and so wasting all their strength upon it, that they have none left in them to draw the corollary, to do good with" (Sterne [1759–67], pp. 87–8).

But I propose to draw that corollary. In the spirit of Hilbert's program, my project is not to provide epistemological foundations for the strongest system I can, starting from the bottom up, only to stop there and advocate a restriction of mathematical methods to those so founded. Rather, with Hilbert, I am interested in a system's ability to refer to itself and thereby to demonstrate properties that it has. Since a system's ability to perform these tasks depends on its strength, a precise study of the phenomenon involves studying systems of different strengths. Weak arithmetic theories admit different arithmetization schemes and therefore perform these tasks in different ways. Consequently they are the natural place to turn to investigate how mathematics speaks for itself.

As I said, the strain of Hilbert's thought that I isolate and develop in this book is not ordinarily associated with Hilbert's program. Nevertheless, I believe that once it is recovered, it complements many other, well-known features of Hilbert's thought. But more glaringly, I believe that it runs directly counter to the ideological position that is often, but erroneously, attributed to Hilbert. In Chapters 1 and 6 I focus on the significance of Hilbert's anti-foundational stance, and try to rethink how his philosophical ideas fit with the historical events that provided their context as well as how they have influenced contemporary philosophy of mathematics.

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Curtis Franks
Frontmatter
More information

Acknowledgments

For such a short book as this, I have a long list of debts.

A great deal of the content of this book was inspired by conversations with colleagues during the last half decade or so. Most obviously, my doctoral committee at the University of California's Department of Logic and Philosophy of Science in Irvine not only educated me in the relevant philosophical history and formal techniques, but helped shaped my own views as I was formulating them. This committee consisted of Aldo Antonelli, Jeffrey Barrett, and Kai Wehmeier. I learned a great deal from them all about how to approach philosophical problems. Aldo's guidance in particular, at every stage, was essential. It is not possible to reconstruct the several ways that his advice has shaped this project.

My debt extends well beyond this circle, though, to other specialists who contributed to my understanding of the various philosophical, mathematical, historical, and linguistic topics that arise in this book. Advice from Michael Detlefsen, Penelope Maddy, Volker Halbach, Neil Delany, Richard Grandy, Sam Buss, Richard Rorty, and Thomas Saine has been particularly helpful. Again, the scarcity of specific acknowledgments to these colleagues should be understood as evidence of the depth of their influence.

I owe an equally great debt to those who most encouraged me to turn this project into a book. I share the credit for its publication with Aldo Antonelli, Jeffrey Barrett, Michael Detlefsen, Patricia Blanchette, Richard Rorty, Leo Corry, Andrew Boucher, Hilary Gaskin, and two anonymous reviewers of an earlier manuscript. Without their vision, the book not only would have not taken on the shape that it has taken, but would not likely have appeared in any form.

I am deeply grateful to my students at the University of Notre Dame, who have often been the philosophical audience that pruned and sharpened what I wanted to say. Conversations with Graham Leach-Krouse, Charles

Pence, John Firth, Sean Walsh, Tony Strimple, Jon Buttaci, Louis Gularte, and Alana Stelton were especially valuable.

Among friends I have mentioned twice, both for their contributions to this project and for their vision of its potential, is Richard Rorty, whose advice and optimism, though occasional, were essential. I regret being unable to share this book with him.

I presented an earlier version of Chapter 2 at the *GAP* conference in Berlin in 2006. I regret that for unfortunate legal reasons I was not able to accept Benedict Lowe's invitation to contribute the chapter to the proceedings of his wonderful session, "Towards a new epistemology of mathematics." An earlier version of Chapter 3 was the basis of my talk at the 2008 annual meeting of the Association of Symbolic Logic in Irvine. Chapter 4 grew out of a presentation that I made to a graduate seminar on Gödel that Michael Detlefsen and I organized in 2007, and improvements to Chapter 6 grew out of discussions in my undergraduate seminar "Philosophy against itself" in 2008. I presented an early version of Chapter 5 at the Notre Dame Mathematics Department's Logic colloquium in 2007. I thank the members of these various audiences and seminars, many of whom are not already mentioned above.