CHAPTER 1

A new science

1.1 RECOVERING HILBERT'S THOUGHT

No one disposed to judge the worth of an idea by its impact on culture through contributions to art and science would too quickly dismiss David Hilbert's philosophy of mathematics. Although he worked in an era when mathematicians were especially prone to reflect on the nature of their discipline, when philosophies of mathematics numbered nearly as many as great mathematical minds, the innovative research spawned by Hilbert's unique views stands out for its lasting imprint on mathematical practice. Yet oddly, few people today endorse his views. In the main, they are deplored.

I find this paradox intolerable, and I hope to dissolve it by unearthing its origins. This will be somewhat arduous, but it is worth the effort. Hilbert's ideas have not been rejected because of their faults, but because his true vision is unknown. The excavation that follows will, I hope, expose the genius of his philosophical vision and its essential connection to his mathematical innovations.

In the early twentieth century, Hilbert invented a new formal science – the study of the global properties of branches of mathematics like number theory, analysis, and group theory. This invention allowed one for the first time to investigate in a scientific manner whether, for example, the principles used by analysts are consistent or whether, to take another example, the principles used by group theorists suffice to answer all questions about groups. Hilbert's vision involved two steps. First he explained how to isolate the definitive principles of a branch of mathematics and design a formal system (a set of axioms and inference rules) that embodies those principles. Then he explained how reasoning mathematically about the combinatorial properties of these formal systems leads to the discovery of facts about the entire class of theorems that can be proved with them. Hilbert sometimes called this science "proof theory," other times "meta-mathematics."

A century later, meta-mathematics is a thriving discipline. In addition to the continued interest in questions that have arisen directly out of its development, meta-mathematics has shed insight into some of the deepest problems in computer science, logic, and main-line mathematical research. Hilbert's pioneer efforts in the field have thus proved to be a great achievement. If the significance and beauty of a science speaks for the validity of the ideas that it grew out of, then the views that spawned his efforts have been emphatically vindicated. They bear the rare mark of philosophical genius, the customs stamp signifying their safe landing on science's soil.

One would like to understand Hilbert's philosophical views, to gain some insight into the formative moments of an exciting modern science, even to align one's own understanding of mathematics with his. However, two attitudes dominate contemporary discussion of Hilbert's thought, and their influence screens from historical access the revolutionary insight that led him to forge his new science. The more dominant of these attitudes was probably most forcefully voiced by Alfred Tarski. As early as 1930 Tarski emphasized that the establishment of meta-mathematics as an independent branch of mathematics liberated it from its conceptual origins. Meta-mathematical concepts, he explained in [1931], "do not differ at all from other mathematical notions" so that "their study remains entirely within the domain of normal mathematical reasoning" (p. 111). Specifically he declared that Hilbert's alleged hope that meta-mathematics would usher in a "feeling of absolute security" was "a kind of theology" that lay "far beyond the reach of any normal human science" ([1995], p. 160). Since Hilbert's principal accomplishment, as Tarski saw it, was that his efforts had solidified into a normal human science, the conceptual framework that gave rise to them – being arcane and ideological – is both irrelevant to the further pursuit of meta-mathematics and erroneous as a portrayal of the field's nature. On the same grounds, G. H. Hardy began his famous caricature of Hilbert's thought in his 1928 Rouse Ball lecture with these splintering words:

> I find it very necessary to distinguish between Hilbert the philosopher and Hilbert the mathematician. I dislike Hilbert's philosophy quite as much as I dislike that of Brouwer and Weyl, but I see no reason for supposing that the importance of his logic depends in any way on his philosophy. ([1929], p. 1251)

Opposing this Tarskian tenor is a second, more favorable attitude towards Hilbert's thought. Scholars with this attitude think that Hilbert's philosophical views are well worth taking seriously. They see Hilbert holding an irrealist or nominalist view about the nature of mathematics called "formalism," and espousing "finitism," the skeptical view that only a highly restricted class of mathematical techniques are *prima facie* legitimate. Although these scholars disagree about exactly how Hilbert's formalist and finitist views figure into the vision he had set for meta-mathematics,[1] they agree that he intended to provide an epistemological foundation for mathematics

[1] Two points of contention are the place of conservativity results in Hilbert's program and the status (as meaningful or merely instrumental constructions) of ideal elements. Raatikainen [2003], pp. 166–9 and Mancosu [1998a], pp. 159–61 survey these debates.

in a way that was informed by these views. Yet they tend also to agree in a certain sense with Tarski's idea that Hilbert's foundational aims have been separated from meta-mathematical research in the very process of that body of research attaining the status of an ordinary human science. Thus despite a deep affinity for foundational matters, Georg Kreisel explains that

> the passage *from* the foundational aims for which various branches of modern logic were originally developed *to* the discovery of areas and problems for which its methods are effective tools . . . did not consist of successive refinements, a gradual evolution by adaptation . . . , but required radical changes of direction, to be compared to evolution by migration. ([1985], p. 139)

Kreisel's metaphorical language suggests that logicians had to shed their foundational aspirations in order to enjoy the full flowering of meta-mathematics as a science.

Clearly, neither attitude is of much help for understanding how meta-mathematics came to be. If the philosophical demands that Hilbert placed on his studies no longer shape meta-mathematics in its mature form, then one must turn elsewhere to find the field's true conceptual origins. But it is disingenuous to deny the tremendous formative impact of Hilbert's early proof-theoretical investigations, however entangled these may have been in a grander program. And worse, if meta-mathematics achieved the status of normal science in the very process of its practitioners shifting their attention away from philosophical goals, then there may be no instructive story of its invention to tell – its chief engineers having merely rescued some *accidental* features of an overly ambitious program by showing that these features could function miraculously on their own.

These scholarly attitudes towards Hilbert's philosophical thought have their own conceptual history, though. They could very well have not developed but for the entrenchment of a well-rehearsed story about how Hilbert's chief aims were dashed by two of the earliest meta-mathematical results, Kurt Gödel's incompleteness theorems. According to this story, Hilbert had designed his proof theory in order to demonstrate that the principles of classical mathematics were formally consistent, free from the threat of paradox. Moreover, since this demonstration was supposed to be carried out according to the constraints of formalism and finitism, the ensuing defense of classical mathematics would also serve as evidence that finitary reasoning about formal signs was the ultimate foundation of mathematical activity. Thus the "absolute certainty" that Tarski mocked is to be found in the security of this elementary form of reasoning. But since the incompleteness theorems demonstrate the unavailability of the kind of result Hilbert sought, Hilbert's philosophical views are not only beyond the reach of human science but also clearly erroneous. This story takes on a dramatic tone in the irony it depicts: The incompleteness theorems were among the earliest results to draw significant attention to the then fledgling discipline, so Hilbert's philosophical vision was toppled by the same blow that hammered his technical program into a permanent science.

Seen as reactions to this story, the current opinions about Hilbert's thought begin to make sense. Those sympathetic to formalism and finitism as philosophical doctrines happily let metamathematics continue on its course uninformed by such scruples, embracing an attitude satirized by Richard Rorty:

> In every generation, brilliant and feckless philosophical naifs . . . turn from their own specialties to expose the barrenness of academic philosophy and to explain how some or all of the old philosophical problems will yield to insights gained outside philosophy – only to have the philosophy professors wearily explain that nothing has changed at all. ([1976], p. 32)

Since meta-mathematics left its philosophical roots behind in its passage to normal science, those roots cannot be condemned by results, like Gödel's, issuing from meta-mathematics itself. Finitism and formalism live on, and it is up to philosophical deliberation, not the tribunal of mathematical theorems, to determine their ultimate merits or faults! Meanwhile the less sympathetic, Tarski among them, were more drawn to the elegance of the emerging science than they were concerned with the fate of its conceptual origins. If Hilbert's philosophical views had not been undermined by the verdict issued by his own invention – if they *cannot* be undermined because they are too philosophically pure to need to answer to something as mundane as a scientific result – then foundationalism is merely modern logic's quaint, embarrassing heritage. Hilbert invented meta-mathematics, but its enduring self-image derives from Tarski!

This situation poses an inevitable question: If Hilbert's philosophical ideas were so bad, how could they have been so scientifically productive? For all the compelling drama it depicts, the suggestion that a mathematical discipline should grow out of revolutionary insight, only for later development of the new mathematics to expose that insight's hopelessness and error, is surely implausible. Neither is it any help to try to salvage Hilbert's vision from the wreckage by spiriting it away into philosophically pure, closed-off quarters. That only compounds the mystery. Ideas that are scientifically inert by design ought to yield even less fruit than mistaken but application-oriented ones.

The fact that the attitudes canvassed above lead so naturally to this question explains the difficulty in understanding Hilbert's meta-mathematical revolution. But the dilemma this question poses is no reason to despair. The question is unanswerable because the "bad ideas" that one cannot envision hooking up with their mathematical fruits are distortions of Hilbert's actual thought. In stark contrast to the interpretation that attributes these ideas to him, I see Hilbert's genius stemming precisely from the fact that his ideological inclination was whole-heartedly scientific. His philosophical strength was not in his ability to carve out a position among others about the nature of mathematics, but in his realization that the mathematical techniques already in place suffice to answer questions *about* those techniques – questions that rival thinkers had assumed were the exclusive province of pure philosophy. The conceptual framework that Hilbert continuously referenced in his early proof-theoretical writings was not a hermetic landscape, to be evaluated by the sublimity of the arguments in its favor. Hilbert designed his research program to strip the fate of mathematics from the edict of those who would pronounce on its "true nature" and redeposit it in the hands of the scientific community. To understand the subtlety of his ideas and the way meta-mathematics emerged from them, one must count Hilbert among Rorty's "philosophical naifs." One must see him deliberately offering mathematical explanations where philosophical ones were wanted. He did this, not to provide philosophical foundations, but to liberate mathematics from any apparent need for them. "Defending" Hilbert's ideas by claiming that they are untouched by scientific findings insults that vision. It is, to borrow another of Rorty's quips, "like complimenting a judge on his wise decision by leaving him a fat tip" or like trying to praise a postmodernist by telling him that his views exhibit all of modernity's signs of truth ([1979], p. 372). The legitimacy of Hilbert's philosophical stance lies precisely in its ability to generate an arena for the scientific study of mathematics. Thus the above question inverts. Since Hilbert's philosophical ideas have been so scientifically productive, they must have been quite good. What were they?

To answer *this* question, one must shed the image of Hilbert as a dogmatic finitist and formalist. Then a fresh look at Hilbert's views, as he and as his philosophical colleague and spokesman Paul Bernays explain them in a series of essays and scientific reports in the 1920s, reveals three things. First, it shows how different Hilbert's thinking was from that of his ideological adversaries. It also shows that his thinking was remarkably different from the image that, as a result of a century of rhetoric in line with Tarski's statements, has become a fixture in discussions of the philosophy of mathematics – the image of Hilbert trying to ground Cantor's paradise in the safe turf of finitary reasoning. Most importantly, it uncovers the astonishing depth of Hilbert's philosophical thought. I present this fresh look at Hilbert's views in Chapter 2. In Chapter 6 I return to their appraisal and try to relocate Hilbert on the map of philosophers of mathematics. In the intervening chapters, I sketch the link between Hilbert's philosophical thought and the development of meta-mathematics. There, two morals surface. First, the formal science that Hilbert invented is neither an accidental by-product of some bad ideas nor a philosophically inert discipline whose conceptual origins are forever closed off from study. Rather, the principal techniques of meta-mathematics emerge directly from Hilbert's philosophical vision. Second, modern logic risks steering off a promising course if its practitioners lose sight of this fact. The flexibility of meta-mathematics continues to offer logicians ways to bring mathematical techniques to bear on questions about how and why those very techniques work, just as Hilbert proposed.

But before the recovery and development of Hilbert's views can commence, some of its prehistory is in order. This is the story about nineteenth-century mathematical creativity engendering the "epistemological crises" that led so many mathematicians to feel a need to espouse philosophical views about their science in the first place. The story is a largely familiar one, except that in the place of a single looming crisis I find several competing ones. The first step in understanding Hilbert is identifying the crisis that he was reacting to.

1.2 FREEDOM FROM NATURE

What today goes by the name "pure mathematics," the nineteenth-century German mathematician Georg Cantor called "free mathematics."[2] "Mathematics," he wrote, "is entirely free in its development and its concepts are restricted only by the necessity of being non-contradictory and coordinated to concepts previously introduced by previous definitions" ([1883], p. 896).

Prior to the nineteenth century,[3] from the advent of the scientific revolution, the disciplinary distinction between mathematical and empirical sciences familiar today was unknown. In the words of Penelope Maddy, "[t]he great thinkers of that time – from Descartes and Galileo to Huygens and Newton – did mathematics as science and science as mathematics without any effort to separate the two" ([2008], p. 17). In their able hands, it was a potent mixture. Both the physical and mathematical contributions that hindsight picks out from their work are formidable. But by the late 1800s, the mathematical vanguard considered any tendency to blur the line

[2] In [1883] Cantor emphasized the philosophical significance of his preferred terminology: He wrote that mathematicians are under "*absolutely no* obligation to examine their [ideas'] transient reality" and that "[b]ecause of this remarkable feature – which distinguishes mathematics from all other sciences and provides an explanation for the relatively easy and unconstrained manner with which one may operate with it – [mathematics] especially deserves the name of *free mathematics*, a designation which, if I had the choice, would be given precedence over the now usual 'pure' mathematics" (p. 896).

[3] The story of shifts in mathematical thought between the seventeenth and twentieth centuries has been told many times. My retelling of it follows Morris Kline's exemplary historical work in Chapters 41, 43, and 51 of *Mathematical Thought from Ancient to Modern Times*.

separating mathematical and empirical investigations ideologically crippling. To *these* great thinkers, the most exciting and promising mathematical novelties were predominantly ones that had no motivation or correlate in nature. Whiggish insistence from their conservative colleagues, like Fourier, that "[t]he profound study of nature is the most fertile source of mathematical discoveries," that "[t]he fundamental ideas are those which represent the natural happenings" was to them a nuisance.[4] Mathematicians had endured criticism directed against their growing registry of "unnatural" preoccupations – negative numbers, complex numbers, non-commutative quaternions, non-Euclidean and multi-dimensional spaces – until the sheer bulk of these inventions (and the fact that even their critics freely utilized them in their research) began to make that criticism sound tired and monotonous. According to Morris Kline, "after about 1850, the view that mathematics can introduce and deal with arbitrary concepts and theories that do not have any immediate physical interpretation ... gained acceptance" ([1972], p. 1031). To Cantor, as to many of the mathematicians most immersed in the farthest reaches of abstraction mathematics had to offer, this view was not only acceptable but definitive of their discipline's very nature. "The essence of mathematics," he wrote, "lies in its freedom" ([1883], p. 896).

With freedom from the restrictions of the empirical world came a license for unbridled creativity. Mathematicians came to see their work as an essentially creative activity. As a result, they began to see human genius not only as necessary in order to conquer mathematical terrain, but also as somehow constitutive of that terrain. Cantor's procession of extravagantly infinite sets was deservedly the

4 The passages are from the preface of Fourier's *Analytical Theory of Heat* as they are quoted in Kline [1972], pp. 1036–7.