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GLOBAL METHODS FOR COMBINATORIAL ISOPERIMETRIC PROBLEMS

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11 J.L. Alperin *Local representation theory*
12 P. Koosis *The logarithmic integral I*
13 A. Pietsch *Eigenvalues and s-numbers*
14 S.J. Patterson *An introduction to the theory of the Riemann zeta-function*
15 H.J. Baues *Algebraic homotopy*
16 V.S. Varadarajan *Introduction to harmonic analysis on semisimple Lie groups*
17 W. Dicks & M. Dunwoody *Groups acting on graphs*
18 L.J. Corwin & F.P. Greenleaf *Representations of nilpotent Lie groups and their applications*
19 R. Fritsch & R. Piccinini *Cellular structures in topology*
20 H. Klingen *Introductory lectures on Siegel modular forms*
21 P. Koosis *The logarithmic integral II*
22 M.J. Collins *Representations and characters of finite groups*
24 H. Kunita *Stochastic flows and stochastic differential equations*
25 P. Wojtaszczyk *Banach spaces for analysts*
26 J.E. Gilbert & M.A.M. Murray *Clifford algebras and Dirac operators in harmonic analysis*
27 A. Frohlich & M.J. Taylor *Algebraic number theory*
28 K. Goebel & W.A. Kirk *Topics in metric fixed point theory*
29 J.E. Humphreys *Reflection groups and Coxeter gropus*
30 D.J. Benson *Representations and cohomology I*
31 D.J. Benson *Representations and cohomology II*
32 C. Allday & V. Puppe *Cohomological methods in transformation groups*
33 C. Soulé et al *Lectures on Arakelov geometry*
34 A. Ambrosetti & G. Prodi *A primer of nonlinear analysis*
35 J. Palis & F. Takens *Hyperbolicity, stability and chaos at homoclinic bifurcations*
37 Y. Meyer *Wavelets and operators I*
38 C. Weibel *An introduction to homological algebra*
39 W. Bruns & J. Herzog *Cohen-Macaulay rings*
40 V. Snaith *Explicit Brauer induction*
41 G. Laumon *Cohomology of Drinfield modular varieties I*
42 E.B. Davies *Spectral theory and differential operators*
43 J. Diestel, H. Jarchow & A. Tonge *Absolutely summing operators*
44 P. Mattila *Geometry of sets and measures in euclidean spaces*
45 R. Pinsky *Positive harmonic functions and diffusion*
46 G. Tenenbaum *Introduction to analytic and probabilistic number theory*
47 C. Peskine *An algebraic introduction to complex projective geometry I*
48 Y. Meyer & R. Coifman *Wavelets and operators II*
49 R. Stanley *Enumerative combinatorics I*
50 I. Porteous *Clifford algebras and the classical groups*
51 M. Audin *Spinning tops*
52 V. Jurdjevic *Geometric control theory*
53 H. Voelklein *Groups as Galois groups*
54 J. Le Potier *Lectures in vector bundles*
55 D. Bump *Automorphic forms*
56 G. Laumon *Cohomology of Drinfeld modular varieties II*
57 D.M. Clarke & B.A. Davey *Natural dualities for the working algebraist*
59 P. Taylor *Practical foundations of mathematics*
60 M. Brodmann & R. Sharp *Local cohomology*
61 J.D. Dixon, M.P.F. Du Sautoy, A. Mann & D. Segal *Analytic pro-p groups, 2nd edition*
62 R. Stanley *Enumerative combinatorics II*
64 J. Jost & X. Li-Jost *Calculus of variations*
68 Ken-iti Sato *Lévy processes and infinitely divisible distributions*
71 R. Blei *Analysis in integer and fractional dimensions*
72 F. Borceux & G. Janelidze *Galois theories*
73 B. Bollobás *Random graphs*
74 R.M. Dudley *Real analysis and probability*
75 T. Sheil-Small *Complex polynomials*
76 C. Voisin *Hodge theory and complex algebraic geometry I*
77 C. Voisin *Hodge theory and complex algebraic geometry II*
78 V. Paulsen *Completely bounded maps and operator algebras*
79 F. Gesztesy & H. Holden *Soliton equations and their algebro geometric solutions I*
81 S. Mukai *An introduction to invariants and moduli*
82 G. Toutlakis *Lectures in logic and set theory I*
83 G. Toutlakis *Lectures in logic and set theory II*
84 R.A. Bailey *Association schemes*
91 I. Moerdijk & J. Mrcun *Introduction to foliations and lie groupoids*

Global Methods for Combinatorial Isoperimetric Problems

Certain constrained combinatorial optimization problems have a natural analogue in the continuous setting of the classical isoperimetric problem. The study of so-called combinatorial isoperimetric problems exploits similarities between these two, seemingly disparate, settings. This text focuses on global methods. This means that morphisms, typically arising from symmetry or direct product decomposition, are employed to transform new problems into more restricted and easily solvable settings whilst preserving essential structure.

This book is based on Professor Harper's many years experience in teaching this subject and is ideal for graduate students entering the field. The author has increased the utility of the text for teaching by including worked examples, exercises and material about applications to computer science. Applied systematically, the global point of view can lead to surprising insights and results and established researchers will find this to be a valuable reference work on an innovative method for problem solving.

Global Methods for Combinatorial Isoperimetric Problems

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Dedicated to the memory of Edward C. Posner, friend and mentor

Contents

Preface

The purpose of this monograph is a coherent introduction to global methods in combinatorial optimization. By "global" we mean those based on morphisms, i.e. maps between instances of a problem which preserve the essential features of that problem. This approach has been systematically developed in algebra, starting with the work of Jordan in 1870 (see [**90**]). Lie's work on continuous groups, which he intended to apply to differential equations, and Klein's work on discrete groups and geometry (the Erlanger program) resulted from a trip the two made to Paris where they were exposed to Jordan's ideas. Global methods are inherent in all of mathematics, but the benefits of dealing with morphisms do not always justify the effort required and it has also been ignored in many areas. This has been especially true of combinatorics which is viewed by most of its practitioners as the study of finite mathematical structures, such as graphs, posets and designs, the focus being on problem-solving rather than theory-building.

What kinds of results can global methods lead to in combinatorics? Notions of symmetry, product decomposition and reduction abound in the combinatorial literature and these are by nature global concepts. Can we use the symmetry or product decomposition of a particular combinatorial problem to systematically reduce its size and complexity? Many of our results give positive answers to this question. We are not claiming, however, that the global point of view is the only valid one. On the contrary, we are endeavoring to show that global methods are complimentary to other approaches. Our focus is on global methods because they present opportunities which still remain largely unexploited.

The history of mathematics shows that "point of view" can be very important. What is difficult from one point of view may become easy from another. The classical Greek problems of constructing tangents for a plane curve and calculating the area enclosed by such curves were effectively solved only after the introduction of Cartesian coordinates. This allowed geometry to be translated into algebra, from which the patterns of the solutions sprang forth, creating calculus. Generally, the more varied and effective the points of view which a subject admits, the more profound and useful it becomes. It is our contention that the global point of view is effective for at least two of the most important problems of combinatorial optimization, namely the minimum path problem and the maximum flow problem (see [**6**], or one of the many other books on algorithmic analysis, which will verify the predominance of these two problems). It has made some results easier to discover, easier to prove, easier to communicate and teach, and easier to generalize. This monograph was written to demonstrate the validity of these claims for isoperimetric problems on graphs, a subject closely related to the minimum path problem.

In this monograph, morphisms are used to represent reductions (simplifications) of a problem. Such a morphism typically maps a structure, representing an instance of a problem, to another structure of the same kind, in such a way that the essence of the problem is preserved. The latter structure, the *range* of the morphism, is typically smaller than the former, the *domain*. But since a morphism "preserves" the problem, solving it on the range will give the solution for the domain. In this volume we only deal with morphisms in an elementary way so there is no need to use (or even know) category theory. The reader should be aware, however, should questions arise, that category theory is the road map of morphism country.

The term "combinatorial (or graphical) isoperimetric problem" is now part of the language of combinatorics, but its first use, 35 years ago in the title of [**46**], was intended to be somewhat shocking. The classical isoperimetric problem of Greek geometry is inherently continuous, involving notions of area, and length of boundary, whereas combinatorial structures are finite and inherently discontinuous. How can they go together? The apparent oxymoron was applied to graphs and posets in an effort to draw attention to an analogy between certain natural constrained optimization problems on those structures and the classical isoperimetric problem of Euclidean geometry. Initially, the nomenclature was meant to reinforce the idea that these combinatorial problems were fundamental and therefore deserving of further study. As we see in this text and others, the analogy has also shown the way for adapting powerful algebraic and analytic tools from classical mathematics to solve combinatorial problems.

There are three classes of previous publications (monographs) which relate to this one:

(1) Surveys of combinatorial optimization, including isoperimetric problems, focusing on results:
   (a) *Sperner Theory in Partially Ordered Sets* by K. Engel and H.-D. O. F. Gronau (1985) **[35]**

   (b) *Combinatorics of Finite Sets* by I. Anderson (1987) **[7]**
   (c) *Sperner Theory* by K. Engel (1997) **[34]**.
(2) One book which develops global methods, Steiner symmetrization and its variants, for solving (continuous) isoperimetric problems arising in applications: *Isoperimetric Inequalities in Mathematical Physics* by G. Polya and G. Szegö (1951) **[84]**.
(3) Research monographs which develop discrete analogs of harmonic and spectral analysis to solve combinatorial problems related to isoperimetry:
   (a) *Group Representations in Probability and Statistics* by P. Diaconis (1988) **[33]**
   (b) *Discrete Groups, Expanding Graphs and Invariant Measures* by A. Lubotzky (1994) **[75]**
   (c) *Spectral Graph Theory* by F. R. K. Chung (1997) **[27]**.

This volume may be thought of as in the same spirit as the monographs of (3), taking the combinatorial isoperimetric problems of (1) and developing the discrete analogs of the global methods of (2) to solve them. The reality behind it, however, was a bit more complex. Compression was already a standard tool of combinatorialists in the mid-1970s when the author asked the questions (above, second paragraph) leading to the development of stabilization. In about 1976 G.-C. Rota, an outstanding mathematical scholar as well as one of the best listeners of the author's acquaintance, remarked on the analogy between stabilization and Steiner symmetrization, referring to the Polya–Szegö monograph. Even though the definitions of stabilization and compression are simple and natural in the combinatorial context, it seems unlikely that they would have been found by anyone deliberately seeking a discrete analog of symmetrization. Furthermore, the combinatorial setting leads naturally to the definition of partial orders, called stability and compressibility orders in this monograph, which characterize stable or compressed sets as ideals. This goes beyond the Polya–Szegö theory and it is not yet clear how to define the corresponding structures in the continuous context. However Steiner's historical model of global methods, with its wealth of applications, has given guidance and the assurance of depth to the combinatorial project.

It is probably too early yet to make a definitive statement about how global methods stack up against harmonic and spectral analysis (harmonic analysis may be identified with the spectral analysis of a Laplacian) as they are all still being developed and applied. They each have historical roots in the early nineteenth century: harmonic analysis beginning with Fourier, spectral analysis with Sturm–Liouville and global methods with Steiner (symmetrization). The author first confronted the question of how they relate in 1969: Having solved the wirelength and bandwidth problems on the d-cube where the objective functions are L_1 and L_∞ norms respectively, it was natural to pose the L_2 analog but the combinatorial methods which had been successful in L_1 and L_∞ did not go very far in L_2. Several years later a beautiful solution to the L_2 problem, using harmonic analysis on the group 2^d, was published by Crimmins, Horwitz, Palermo and Palermo [**30**]. On the other hand, harmonic analysis on the group 2^d did not seem capable of matching the results produced by global methods on the L_1 and L_∞ problems, so we concluded that the two approaches are complimentary, each majorizing the other on different problems. Recently, a superficial examination of the evidence which has accumulated since 1969 led to the same conclusion. It would seem fair to say that stabilization, the Steiner operation based on symmetry, is very limited compared to harmonic analysis. After all, it only works for reflective symmetry of a special kind. But for many of the problems where it does apply (and there are a number of important ones), the results from stabilization are more accurate and seem likely to remain so.

One of the most exciting prospects for applications of isoperimetric inequalities in recent years is the connection with the rate of convergence of a random walk, the focus of Diaconis's monograph [**33**]. After Diaconis calculated that it takes seven riffle shuffles to randomize a deck of cards, it became a legal requirement for black jack dealers in Las Vegas. The same mathematics is at the foundation of efficient random algorithms for many problems which would otherwise be intractable.

Applications are the touchstone of mathematics. The author started solving combinatorial isoperimetric problems as a research engineer in communications at the Jet Propulsion Laboratory. Since then, as a mathematician at the Rockefeller University and the University of California at Riverside, applications to science and engineering have continued to motivate the work. A good application for the solution of a hard problem doubles the pleasure, and every other benefit, from it. Global methods are by nature abstract and might easily degenerate into what von Neumann called "baroque mathematics" if not guided by real applications. On several occasions over the years, promising technical insights were left undeveloped until the right application came along. We would recommend that same caution to others developing global methods.

This monograph grew out of lectures given in the graduate combinatorics course at the University of California, Riverside from 1970 to the present (2003). The first five chapters develop the core concepts of the theory and have been pretty much the same since 1990. The development is pedestrian, assumes only an elementary knowledge of combinatorics, and largely follows the logic of discovery:

Chapter 1 is preglobal: Defining the edge-isoperimetric problem, solving it for the d-cube and presenting several applications to engineering problems.

Chapter 2 is transitional: Bringing out the connection between the edge-isoperimetric problem and the minimum path problem on networks and observing that the minimum path problem has a natural notion of morphism which extends to the edge-isoperimetric problem.

Chapter 3 is central: Defining stabilization and compression, developing their basic theory and demonstrating its efficacy in solving edge-isoperimetric problems.

Chapter 4 reinforces 1–3: Defining the vertex-isoperimetric problem and showing how stabilization and compression are also effective on it.

Chapter 5: Begins the process of deepening and extending the theory of stabilization, mainly by connecting it with Coxeter's theory of groups generated by reflections.

Those first five chapters have been used repeatedly (the course was generally offered every second or third year) for a one-quarter graduate course. The last five chapters extend the core material of Chapter 3 in five different directions. Since our thesis is that systematizing and refining the ideas that had solved challenging combinatorial isoperimetric problems would open up new possibilities, demonstrations were required to make the case. The last five chapters, which resulted from research based on the first five, constitute the necessary demonstrations.

Chapter 6: Begins the process of deepening and extending the theory of compression.

Chapter 7: Extends the theory of stabilization to infinite graphs.

Chapter 8: Extends the isoperimetric problems and their global methods to higher dimensional complexes (hypergraphs).

Chapter 9: Builds on the results of chapters 3, 4, 5 and 6 which show that isoperimetric problems on graphs can, in many interesting cases, be reduced to maximizing the weight of an ideal of fixed cardinality in a weighted poset. A new notion of morphism for this maximum weight ideal problem is introduced and applied to solve several combinatorial isoperimetric problems which seemed impossibly large.

Chapter 10: Reintroduces one of the oldest tools of optimization: calculus. The main combinatorial tool for solving isoperimetric problems on infinite families of graphs is compression. Compression requires that all members of the family have nested solutions and this is not the case for some isoperimetric problems that arise frequently in applications. Passage to a continuous limit which may then be solved by analytic methods can give asymptotic solutions, and in some cases even exact solutions. In the most interesting cases passage to the limit is facilitated by stabilization.

The last five chapters could be covered in a second quarter course. A semester course should cover the first five chapters and then a selection of the last five as time allows. Some of them presume a bit of background in algebra or analysis. The book may also be used for self-study, in which case we recommend that it be studied top-down, rather than front-to-back. If a concept does not sit well, look at its variants and analogs as they occur elsewhere in the text. That was, after all, how the material developed in the first place.

I wish to thank my students over the years for their effort, patience and helpful feedback, especially Joe Chavez who prepared a preliminary version of the manuscript. Conversations with Sergei Bezrukov, Konrad Engel and Gian-Carlo Rota have also had a profound effect on my thinking about the subject.

Working on combinatorial isoperimetric problems, from the summer of 1962 to the present, has been the greatest aesthetic experience of my life. Coxeter, in his classic monograph on regular polytopes [**28**], points out that the theory of groups generated by reflections (which underpins our concept of stabilization), is also the mathematical basis of the kaleidoscope. The word kaleidoscope is derived from Greek which translates as "beautiful thing viewer." I hope that some of those beautiful things are visible in this presentation.