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Matroid Bundles

LAURA ANDERSON

ABSTRACT. Combinatorial vector bundles, or *matroid bundles*, are a combinatorial analog to real vector bundles. Combinatorial objects called *oriented matroids* play the role of real vector spaces. This combinatorial analogy is remarkably strong, and has led to combinatorial results in topology and bundle-theoretic proofs in combinatorics. This paper surveys recent results on matroid bundles, and describes a canonical functor from real vector bundles to matroid bundles.

1. Introduction

Matroid bundles are combinatorial objects that mimic real vector bundles. They were first defined in [MacPherson 1993] in connection with *combinatorial differential manifolds*, or *CD manifolds*. Matroid bundles generalize the notion of the "combinatorial tangent bundle" of a CD manifold. Since the appearance of McPherson's article, the theory has filled out considerably; in particular, matroid bundles have proved to provide a beautiful combinatorial formulation for characteristic classes.

We will recapitulate many of the ideas introduced by McPherson, both for the sake of a self-contained exposition and to describe them in terms more suited to our present context. However, we refer the reader to [MacPherson 1993] for background not given here. We recommend the same paper, as well as [Mnëv and Ziegler 1993] on the combinatorial Grassmannian, for related discussions.

We begin with a key intuitive point of the theory: the notion of an oriented matroid as a combinatorial analog to a vector space. From this we develop matroid bundles as a combinatorial bundle theory with oriented matroids as fibers. Section 2 will describe the category of matroid bundles and its relation to the category of real vector bundles. Section 3 gives examples of matroid bundles arising in both combinatorial and topological contexts, and Section 4 outlines some of the techniques that have been developed to study matroid bundles.

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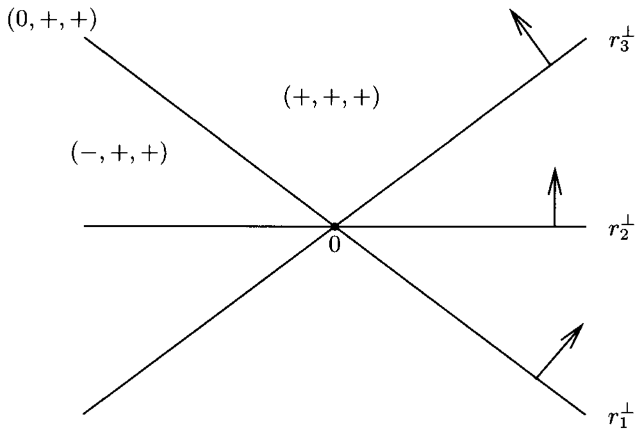

Figure 1. An arrangement of oriented hyperplanes in R^2 and some of the resulting sign vectors.

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1A. Oriented matroids. We give a brief introduction to oriented matroids, particularly to the idea of oriented matroids as "combinatorial vector spaces". See [Björner et al. 1993] for a more complete introduction to oriented matroids, and [MacPherson 1993, Appendix] for specific notions of importance here.

A rank-n oriented matroid can be considered as a combinatorial analog to an arrangement $\{r_i\}_{i\in E}$ of vectors in $\mathbb{R}^n$, or equivalently, to an arrangement $\{r_i^\perp\}_{i\in E}$ of oriented hyperplanes. (Here we allow the "degenerate hyperplane" $0^\perp = \mathbb{R}^n$.) The idea is as follows. An arrangement $\{r_i^\perp\}_{i\in E}$ of oriented hyperplanes in $\mathbb{R}^n$ partitions $\mathbb{R}^n$ into cones. Each cone C can be identified by a sign vector $v \in \{-,0,+\}^E$, where v_i indicates on which side of $r_i^\perp$ the cone C lies. (See Figure 1).

The set E together with the collection of sign vectors resulting from this arrangement is called a *realizable oriented matroid.* The sign vectors are called *covectors* of the oriented matroid. Every realizable oriented matroid has 0 as a covector. The hyperplanes describe a cell decomposition of the unit sphere in $\mathbb{R}^n$, with each cell labeled by a nonzero covector.

More generally, an *oriented matroid* M is a finite set E together with a collection $V^*(M)$ of signed sets in $\{-,0,+\}^E$, satisfying certain combinatorial axioms inspired by the case of realizable oriented matroids. (For a complete definition, see [Björner et al. 1993, Section 4.1].) In this more general context, we still have

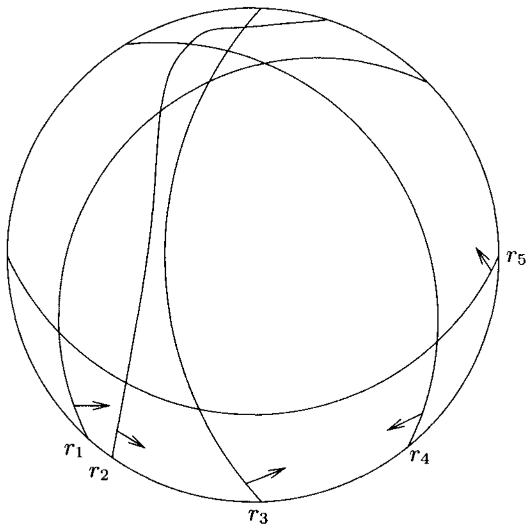

Figure 2. A rank-3 arrangement of five oriented pseudospheres.

a notion of the *rank* of an oriented matroid [MacPherson 1993, Section 5.3], and a beautiful theorem that gives this notion topological meaning.

The *Topological Representation Theorem* of Folkman and Lawrence [Björner et al. 1993, Section 1.4; Folkman and Lawrence 1978] says that the set of nonzero covectors of a rank-n oriented matroid describe a cell decomposition of S^{n-1}. More precisely: a *pseudosphere* in S^{n-1} is a subset S such that some homeomorphism of S^{n-1} takes S to an equator. Thus, a pseudosphere must partition S^{n-1} into two pseudohemispheres. An *oriented pseudosphere* is a pseudosphere together with a choice of positive pseudohemisphere. An *arrangement of oriented pseudospheres* is a set of oriented pseudospheres on S^{n-1} whose intersections behave topologically like intersections of equators. (For a precise definition, see [Björner et al. 1993, Definition 5.1.3].) For an example, see Figure 2.

An arrangement $\{S_i\}_{i\in E}$ of oriented psuedospheres in S^{n-1} determines a collection of signed sets in $\{-,0,+\}^E$ in the same way that an arrangement of oriented hyperplanes in $\mathbb{R}^n$ does. The Topological Representation Theorem states that any collection of signed sets arising in this way is the set of nonzero covectors of an oriented matroid, and that every oriented matroid arises in this way.

1B. Oriented matroids as "combinatorial vector spaces". A *strong map image* of an oriented matroid M is an oriented matroid N such that $V^*(N) \subseteq V^*(M)$. (Strong maps are called *strong quotients* in [Gelfand and MacPherson 1992]. See [Björner et al. 1993, Section 7.7] for more on strong maps.)

Consider a realizable rank-n oriented matroid M, realized as a set $R = \{r_1^\perp, r_2^\perp, \dots, r_m^\perp\} \subset \mathbb{R}^n$. If V is a rank-k subspace in $\mathbb{R}^n$, consider the rank-k oriented matroid $\gamma_R(V)$ given by the intersections $\{V \cap r_i^\perp : i \in \{1,\dots,m\}\}$. In terms of the vector picture of oriented matroids, $\gamma_R(V)$ is given by the orthogonal

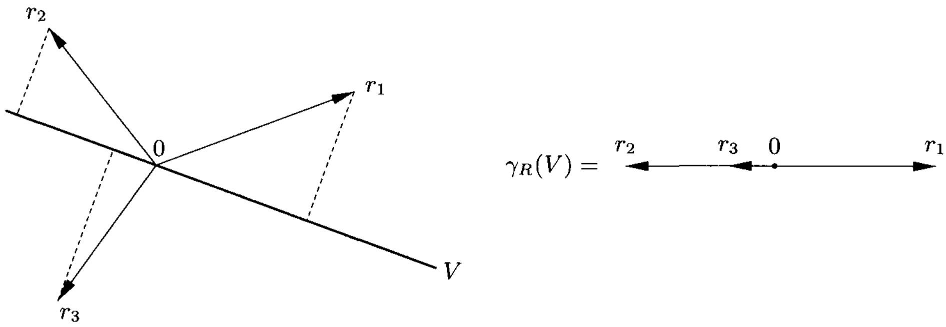

Figure 3. A strong map of realizable oriented matroids.

projections of the elements $\{r_1, \dots, r_m\}$ onto V. The oriented matroid $\gamma_R(V)$ is a strong map image of M, and encodes considerable geometric data about V. For instance, the loops in $\gamma_R(V)$ are exactly those r_i such that $V \subseteq r_i^\perp$, and the cell decomposition of the unit sphere S_V in V given by the equators $S_V \cap r_i^\perp$ is canonically isomorphic to the cell complex of nonzero covectors of $\gamma_R(V)$. We will think of $\gamma_R(V)$ as a combinatorial model for V, and as a combinatorial "subspace" of M. Figure 3 shows a realization of a rank-2 oriented matroid, a 1-dimensional subspace V of $\mathbb{R}^2$, and the resulting oriented matroid $\gamma_R(V)$.

If M is not realizable, we will still use M as a combinatorial analog to $\mathbb{R}^n$, with the nonzero covectors in $V^*(M)$ playing the role of the unit sphere. Strong map images will be viewed as "pseudosubspaces".

1C. Matroid bundles. Consider a real rank-k vector bundle $\xi : E \to B$ over a compact base space. Choose a collection $\{e_1, \dots, e_n\}$ of continuous sections of ξ such that at each point b in B, the vectors $\{e_1(b), \dots, e_n(b)\}$ span the space $\xi^{-1}(b)$. The vectors $\{e_1(b), \dots, e_n(b)\}$ determine a rank-k oriented matroid $M(b)$ with elements the integers $\{1, \dots, n\}$. Note that any $b \in B$ has an open neighborhood U_b such that $M(b')$ *weak maps* to $M(b)$ for all $b' \in U_b$. (See [Björner et al. 1993, Section 7.7] for a definition of weak maps. Weak maps are called *specializations* in [MacPherson 1993] and *weak specializations* in [Gelfand and MacPherson 1992].)

PROPOSITION 1.1. *Let $\xi : E \to B$ be a real vector bundle with B finite-dimensional and let $\mu : |T| \to B$ be a triangulation of B. Then there exists a simplicial subdivision T' of T and a spanning collection of sections of ξ such that for every simplex σ of T', the function M is constant on the relative interior of $\mu(|\sigma|)$.*

This is a corollary to the Combinatorialization Theorem in Section 2C.

EXAMPLE. Figure 4 shows the Möbius strip as a line bundle over S^1, and a triangulation of S^1 with vertices a, b, c. The sections $\{\rho_1, \rho_2\}$ associate a single oriented matroid to the interior of each simplex.

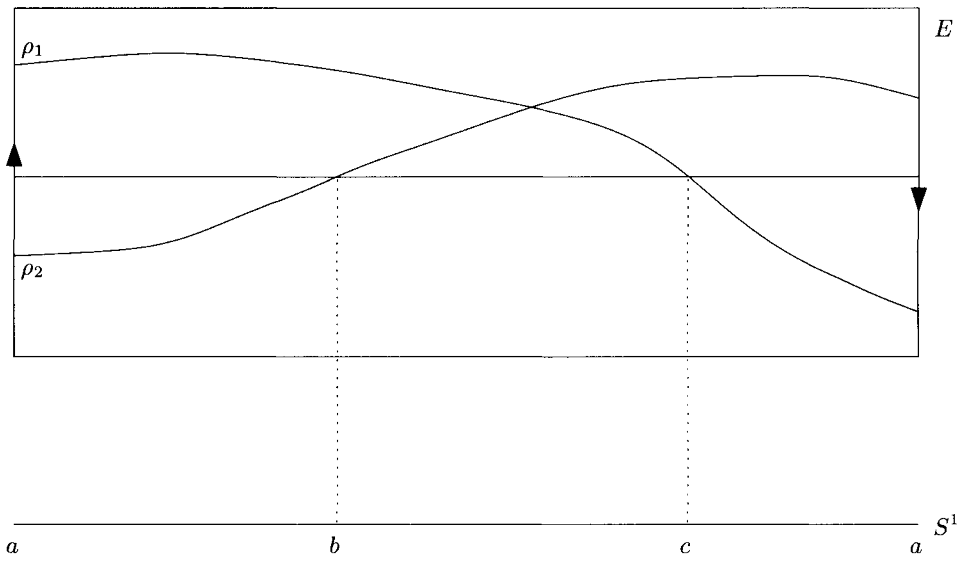

Figure 4. A spanning collection of sections for the Möbius strip.

Such a simplicial complex and the association of an oriented matroid to each cell give the motivating example of a matroid bundle:

DEFINITION 1.2. A rank-k *matroid bundle* is a partially ordered set B (e.g., a simplicial complex with simplices ordered by inclusion) and a rank-k oriented matroid $\mathcal{M}(b)$ associated to each element b, so that $\mathcal{M}(b)$ weak maps to $M(b')$ whenever $b \geq b'$.

This is a simplification of the definition which appears in [MacPherson 1993]. Any matroid bundle in the sense of MacPherson gives a matroid bundle in the present sense. Conversely, given a matroid bundle $(B, \mathcal{M})$ in our current sense, consider the *order complex* ΔB of B, i.e., the simplicial complex of all chains in the partial order. The map associating to each simplex $b_1 \leq \cdots \leq b_m$ in ΔB the oriented matroid $\mathcal{M}(b_m)$ defines a matroid bundle in the sense of MacPherson.

A matroid bundle need not arise from a real vector bundle. For instance, a matroid bundle may include non-realizable oriented matroids as fibers. Section 3 will give examples of matroid bundles arising in combinatorics that do not correspond to any real vector bundles.

1D. What do we want from matroid bundles? The hope is that the category of matroid bundles is closely related to the category of real vector bundles, or perhaps to one of its weaker cousins, such as the category of piecewise-linear microbundles or the category of spherical quasifibrations. (These categories are described below.) Relating bundle theory to oriented matroids promises both combinatorial techniques for bundle theory and bundle-theoretic techniques for combinatorics.

We describe these categories very briefly here. Good sources for a more extended look at bundles are [Milnor and Stasheff 1974; Husemoller 1996]. The loose idea is as follows: a (topological) *bundle* is a map $\xi : E \to B$ of topological spaces such that for some open cover $\{U_i\}_{i \in I}$ of B, each restriction $\xi|_{\xi^{-1}(U_i)}$ "looks like" a projection $p : U_i \times F \to U_i$, for some space F. Different bundle theories arise from different notions of "looking like a projection". E is the *total space* of the bundle, and B is the *base space*. For any $b \in B$, the preimage $\xi^{-1}(b)$ is the *fiber* of ξ over b. A *morphism* from a bundle $\xi_1 : E_1 \to B_1$ to a bundle $\xi_2 : E_2 \to B_2$ is a commutative diagram

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ {\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ B_1 & \longrightarrow & B_2 \end{array}$$

such that the map of total spaces preserves appropriate structure on fibers.

Three progressively weaker categories of bundles are of particular interest. The strongest is the category **Bun** of *real vector bundles*, in which $F \cong \mathbb{R}^k$ and for each U_i we must have a homeomorphism $h : U_i \times \mathbb{R}^k \to \xi^{-1}(U_i)$ such that

$$\begin{array}{ccc} U_i \times \mathbb{R}^k & \xrightarrow{\ h\ } & \xi^{-1}(U_i) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \xi} & \\ & U_i & \end{array}$$

commutes and h restricts to a linear isomorphism on each fiber. In the weaker category **PL** of *piecewise-linear microbundles*, F is still $\mathbb{R}^k$, but the maps h need only be piecewise-linear homeomorphisms with compatible 0 cross-sections. (See [Milnor 1961] for a precise definition.) A still weaker notion is that of a *quasifibration*, which must only "look like" a projection in that for each $x \in U_i$, $y \in \xi^{-1}(x)$, and $j \in \mathbb{N}$, the map of homotopy groups

$$p_* : \pi_j(p^{-1}(U_i), p^{-1}(x), y) \longrightarrow \pi_j(U_i, x)$$

is an isomorphism; see [Dold and Thom 1958, §§ 1.1, and 2.1]. From this condition it follows that each fiber has the same weak homotopy type. We will be interested in the category **Fib** of quasifibrations whose fibers are homotopy spheres. Any real vector bundle or PL microbundle has a canonical associated sphere bundle — essentially by taking a sphere around 0 in each fiber — which is a spherical quasifibration.

Associated to any good bundle theory is a *universal bundle* — that is, a bundle $\Xi : E_\infty \to B_\infty$ such that

1. for any bundle $\xi : E \to B$ there exists a morphism from ξ to Ξ, and
2. if $\xi_1 : E_1 \to B_1$ and $\xi_2 : E_2 \to B_2$ are bundles and $F : \xi_1 \to \xi_2$, $C_1 : \xi_1 \to \Xi$, and $C_2 : \xi_2 \to \Xi$ are morphisms, then there exists a *bundle homotopy* from

C_1 to $C_2 \circ F$, i.e., a morphism H from $\xi_1 \times \mathrm{id} : E_1 \times I \to B_1 \times I$ to Ξ such that $H|_{\xi_1 \times \{0\}} = C_1 \times *$ and $H|_{\xi_1 \times \{1\}} = (C_2 \circ F) \times *$.

In this situation B_∞ is called a *classifying space* for the category. For any bundle ξ and bundle map from ξ to the universal bundle, the map of base spaces is called a *classifying map*. It follows from the properties above that the universal bundle is unique up to bundle homotopy, and that for a fixed universal bundle and fixed real vector bundle, the classifying map is unique up to homotopy. In fact, every vector bundle over a base space B is characterized up to isomorphism by a homotopy class of maps from B to B_∞. Specifically, a bundle ξ over B with classifying map $c(\xi)$is isomorphic to the *pullback* of Ξ by $c(\xi)$, i.e., the bundle $\pi_1 : \{(b, v) : b \in B,\ v \in \Xi^{-1}(c(\xi)(b))\} \to B$. In this way isomorphism classes of bundles over a space B are in bijection with homotopy classes of maps $B \to B_\infty$. Thus if G_1 and G_2 are two categories of bundles with classifying spaces B^1_∞ and B^2_∞ then any map $B^1_\infty \to B^2_\infty$ gives a functor from isomorphism classes in G_1 to isomorphism classes in G_2.

For rank-k real vector bundles over paracompact base spaces, the classifying space (often called BO_k) is $G(k, \mathbb{R}^\infty)$, the space of all k-dimensional subspaces of $\mathbb{R}^\infty$. The universal bundle is the tautological bundle

$$\begin{array}{rcl} E_\infty = \{(V, x) : V \in G(k, \mathbb{R}^\infty),\ x \in V\} & \longrightarrow & G(k, \mathbb{R}^\infty), \\ (V, x) & \longrightarrow & V. \end{array}$$

(See [Milnor and Stasheff 1974, Chapter 5] for details.) The classifying spaces BPL_k for PL microbundles and BFib_k for spherical quasifibrations are harder to describe explicitly, and we won't attempt it here. (See [Milnor 1961, Chapter 5; Stasheff 1963] for constructions. We note in passing that BFib_k is isomorphic to the classifying space for rank-k spherical fibrations [Stasheff 1963] — see the related discussion in [Anderson and Davis $\geq$ 1999].) Since BO_k has a natural PL microbundle structure and BPL_k has an associated spherical quasifibration, there are canonical (up to homotopy) classifying maps $\mathrm{BO}_k \to \mathrm{BPL}_k \to \mathrm{BFib}_k$, giving canonical functors from real vector bundles to PL bundles to spherical quasifibrations.

How do matroid bundles fit into this picture? In Section 2A we will define morphisms of matroid bundles, leading to a category MB_k of matroid bundles. This category has a universal bundle, whose classifying space is called the *MacPhersonian* $\mathrm{MacP}(k, \infty)$. We can relate matroid bundles to other bundle theories by finding nice maps between $\mathrm{MacP}(k, \infty)$ and other classifying spaces.

Topologically, the category of matroid bundles is awkward in that the fibers are combinatorial objects — oriented matroids — which form no topological total space. In Section 4 we will discuss how the Topological Representation Theorem allows us to associate a spherical quasifibration (easily) and even a PL microbundle (gruelingly) to a matroid bundle, giving maps $\mathrm{MacP}(k, \infty) \to \mathrm{BFib}_k$ and $\mathrm{MacP}(k, \infty) \to \mathrm{BPL}_k$ and hence giving functors of bundle theories. Another key result is the Combinatorialization Theorem described in Section 2C, which implies a map $BO_k \to \mathrm{MacP}(k, \infty)$ and another functor.

Much of the progress on matroid bundles has been in the area of *characteristic classes*. A characteristic class for a bundle theory is a rule assigning to each bundle $\xi : E \to B$ an element $u(\xi)$ of $H^*(B)$ such that if

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ {\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ B_1 & \xrightarrow{\;f\;} & B_2 \end{array}$$

is a bundle map, then $u(\xi_1) = f^*(u(\xi_2))$. (See [Milnor and Stasheff 1974] for much more on characteristic classes.) From the definition of universal bundles, it follows that if B_∞ is the classifying space for a bundle theory, then the characteristic classes are in bijection with the elements of $H^*(B_\infty)$. (Note we have not specified coefficients for cohomology: different coefficients give different interesting characteristic classes.) Thus the maps $\mathrm{BO}_k \to \mathrm{MacP}(k, \infty)$, $\mathrm{MacP}(k, \infty) \to \mathrm{BPL}_k$, and $\mathrm{MacP}(k, \infty) \to \mathrm{BFib}_k$ give maps $H^*(\mathrm{BFib}_k) \to H^*(\mathrm{MacP}(k, \infty))$, $H^*(\mathrm{BPL}_k) \to H^*(\mathrm{MacP}(k, \infty))$, and $H^*(\mathrm{MacP}(k, \infty)) \to H^*(\mathrm{BO}_k)$ between the characteristic classes of the respective bundle theories. In various cases (e.g., with $\mathbb{Z}_2$ coefficients) these maps can be shown to be surjective. This gives new results on the topology of $\mathrm{MacP}(k, \infty)$ and connects matroid bundles to the many areas of topology that can be described in terms of characteristic classes.

2. Categories of Matroid Bundles and PL Vector Bundles

2A. The category of matroid bundles. Let B be the poset of cells in a PL cell complex $\mathcal{B}$. Any matroid bundle $(B, \mathcal{M})$ on B induces a canonical matroid bundle structure on the poset of cells of any PL subdivision of $\mathcal{B}$, by associating the oriented matroid $\mathcal{M}(\sigma)$ to each cell in the relative interior of a cell $\sigma \in B$. Two matroid bundles on PL cell complexes are defined to be *equivalent* if there exists a common PL subdivision of the cell complexes such that the resulting matroid bundles on this subdivision are identical.

For B an arbitrary poset, a matroid bundle $(B, \mathcal{M})$ induces a matroid bundle structure $(\Delta B, \mathcal{M}')$ on the cell complex $||\Delta B||$ by defining

$$\mathcal{M}'(\{b_1 < b_2 < \cdots < b_m\}) = \mathcal{M}(b_m).$$

We extend the above notion of equivalence by defining $(B, \mathcal{M})$ to be equivalent to $(\Delta B, \mathcal{M}')$.

DEFINITION 2.1. If $(B_1, \mathcal{M}_1)$ and $(B_2, \mathcal{M}_2)$ are two matroid bundles, a *morphism* from $(B_1, \mathcal{M}_1)$ to $(B_2, \mathcal{M}_2)$ is a pair $(f, [C_f, \mathcal{M}_f])$, where f is a PL map from ΔB_1 to ΔB_2 and $[C_f, \mathcal{M}_f]$ is an equivalence class of matroid bundle structures on the mapping cylinder of f that restrict to structures equivalent to $(B_1, \mathcal{M}_1)$ and $(B_2, \mathcal{M}_2)$ at either end.

The *composition* of a morphism $(f, [C_f, \mathcal{M}_f])$ from $(B_1, \mathcal{M}_1)$ to $(B_2, \mathcal{M}_2)$ and a morphism $(g, [C_g, \mathcal{M}_g])$ from $(B_2, \mathcal{M}_2)$ to $(B_3, \mathcal{M}_3)$ is $(g \circ f, [C_{g\circ f}, \mathcal{M}_{g\circ f}])$, where $\mathcal{M}_{g\circ f}$ is determined by $\mathcal{M}_3$ on the simplices of B_3 and by $\mathcal{M}_f$ on the rest of the cells of $C_{g\circ f}$.

The set of rank-k matroid bundles and their morphisms form a category.

DEFINITION 2.2. A morphism from $(B_1, \mathcal{M}_1)$ to $(B_2, \mathcal{M}_2)$ is an *isomorphism* if there exists a morphism from $(B_2, \mathcal{M}_2)$ to $(B_1, \mathcal{M}_1)$ such that the composition of these maps is the identity morphism.

We get a better relation to the category of rank-k real vector bundles by considering only isomorphism classes of matroid bundles:

DEFINITION 2.3. MB_k will denote the category of isomorphism classes of rank-k matroid bundles and their morphisms.

The classifying space for matroid bundles. MB_k has a classifying space very similar in spirit (and, as we shall later see, in topology) to the classifying space $G(k, \mathbb{R}^\infty)$ for real rank-k vector bundles. Just as $G(k, \mathbb{R}^\infty)$ is the space of all rank-k subspaces of any $\mathbb{R}^n$, the classifying space for MB_k will be the set of all strong map images of any combinatorial model for any $\mathbb{R}^n$.

DEFINITION 2.4. If M^n is a rank-n oriented matroid then define the *combinatorial Grassmannian* $\Gamma(k, M^n)$ to be the poset of all rank-k strong map images of M^n, with the partial order $M_1 \geq M_2$ if and only if M_1 weak maps to M_2.

In some papers the combinatorial Grassmannian is defined to be the order complex $\Delta\Gamma(k, M^n)$ of $\Gamma(k, M^n)$.

The combinatorial Grassmannian was first introduced in [MacPherson 1993] and was the subject of a previous survey article [Mnëv and Ziegler 1993], to which we refer the reader for further discussion.

A particularly useful case is when M^n is the coordinate oriented matroid:

DEFINITION 2.5. Let M_n be the coordinate oriented matroid with elements $\{1, 2, \ldots, n\}$, i.e., the oriented matroid realized by the coordinate hyperplanes in $\mathbb{R}^n$. Then $\Gamma(k, M_n)$ is a *standard combinatorial Grassmannian*, or *MacPhersonian*, denoted $\mathrm{MacP}(k, n)$.

This case is especially important because of a nice alternate description:

PROPOSITION 2.6 [Mnëv and Ziegler 1993]. $\mathrm{MacP}(k, n)$ *is the poset of all rank-k oriented matroids with elements* $\{1, 2, \ldots, n\}$, *ordered by weak maps.*

Note that if M_1 strong maps to M_2 then $\Gamma(k, M_2) \subseteq \Gamma(k, M_1)$ (and hence $\Delta\Gamma(k, M_2)$ is a subcomplex of $\Delta\Gamma(k, M_1)$). In particular:

- If $\{1, \ldots, n\}$ is the set of elements of M, $\Gamma(k, M)$ is a subposet of $\mathrm{MacP}(k, n)$.
- If M_2 is obtained from M_1 by deleting some elements, there is a natural embedding of $\Gamma(k, M_2)$ into $\Gamma(k, M_1)$. In particular, $\mathrm{MacP}(k, n) \hookrightarrow \mathrm{MacP}(k, n+1)$ for any k and n.

Thus the direct limit $\lim_{n\to\infty}\Gamma(k, M^n)$ in the category of posets and inclusions is $\bigcup_n \mathrm{MacP}(k, n)$, denoted $\mathrm{MacP}(k, \infty)$.

We can now rephrase the definition of matroid bundles:

DEFINITION 2.7. A rank-k *matroid bundle* is a poset B and a poset map $\mathcal{M} : B \to \mathrm{MacP}(k, \infty)$.

Modifying definitions appropriately (to accomodate our combinatorial notion of bundles and bundle morphisms), we see:

PROPOSITION 2.8. *The map* $\mathrm{id} : \mathrm{MacP}(k, \infty) \to \mathrm{MacP}(k, \infty)$ *is the universal bundle for* MB_k.

PROOF. A matroid bundle $\mathcal{M} : B \to \mathrm{MacP}(k, \infty)$ determines a simplicial map from ΔB to $\Delta\,\mathrm{MacP}(k, \infty)$, and $\mathcal{M}$ induces a matroid bundle structure on the mapping cylinder, giving a classifying map. If $(f, [C_f, \mathcal{M}_f])$ is a matroid bundle morphism, then $(C_f, \mathcal{M}_f)$ determines a homotopy between the respective classifying maps. □

Thus $\mathrm{MacP}(k, \infty)$ is the classifying space for rank-k matroid bundles.

The cohomology of a poset P is defined to be the cohomology of its order complex ΔP. Thus we have:

COROLLARY 2.9. *The* characteristic classes *for* MB_k with coefficients in R *are the elements of the cohomology ring* $H^*(\Delta\,\mathrm{MacP}(k, \infty); R)$.

The finite combinatorial Grassmannians are of interest in their own right from several perspectives. The spaces $\Delta\Gamma(k, M^n)$ arise as the fibers of a combinatorial Grassmannian bundle in [MacPherson 1993], for instance, and $\Delta\Gamma(n-1, M^n)$ is closely related to the *extension space* $\mathcal{E}(M^n)$ discussed in Section 3.

2B. Relations between the real and combinatorial Grassmannians. We consider more closely the map

$$\gamma_R : G(k, \mathbb{R}^n) \to \Gamma(k, M^n)$$

introduced in Section 1B. The set of preimages of this map give a stratification of $G(k, \mathbb{R}^n)$ which is semialgebraic. This stratification has the property that if the closure of $\gamma_R^{-1}(M_1)$ intersects $\gamma_R^{-1}(M_2)$ then M_1 weak maps to M_2.

By the semi-algebraic triangulation theorem [Hironaka 1975], there exists a triangulation of $G(k, \mathbb{R}^n)$ refining this stratification, giving a simplicial map

$$\tilde{\gamma}_R : G(k, \mathbb{R}^n) \to \Delta\Gamma(k, M^n).$$

(This is described further in [MacPherson 1993] for $\mathrm{MacP}(k, n)$ and in [Anderson and Davis $\geq$ 1999] for more general M^n.) In the direct limit this gives a map $\tilde{\gamma} : G(k, \mathbb{R}^\infty) \to \Delta\,\mathrm{MacP}(k, \infty)$ of classifying spaces, and hence describes a map from the theory of real vector bundles to the theory of matroid bundles.