CHAPTER 1

The Helmholtz Equation

1.0 Introduction

The main ideas relating the symmetry group of a linear partial differential equation and the coordinate systems in which the equation admits separable solutions are most easily understood through examples. Perhaps the simplest nontrivial example that exhibits the features we wish to illustrate is the Helmholtz, or reduced wave, equation

$$(\Delta_2+\omega^2)\Psi(x,y)=0 \tag{0.1}$$

where ω is a positive real constant and

$$\Delta_2\Psi=\partial_{xx}\Psi+\partial_{yy}\Psi.$$

(Here $\partial_{xx}\Psi$ is the second partial derivative of Ψ with respect to x.)

In this chapter we will study the symmetry group and separated solutions of (0.1) and related equations in great detail, thereby laying the groundwork for similar treatments of much more complicated problems in the chapters to follow.

For the present we consider only those solutions Ψ of (0.1) which are defined and analytic in the real variables x,y for some common open connected set $\mathfrak{D}$ in the plane R^2. (For example, $\mathfrak{D}$ can be chosen as the plane itself.) The set of all such solutions Ψ forms a (complex) vector space $\mathfrak{F}_0$; that is, if $\Psi\in\mathfrak{F}_0$ and $a\in\mathcal{C}$, then $(a\Psi)(x,y)\equiv a\Psi(x,y)\in\mathfrak{F}_0$, and $(\Psi_1+\Psi_2)(x,y)\equiv\Psi_1(x,y)+\Psi_2(x,y)\in\mathfrak{F}_0$ whenever $\Psi_1,\Psi_2\in\mathfrak{F}_0$. Considering $\mathfrak{D}$ as fixed throughout our discussion, we call $\mathfrak{F}_0$ the *solution space* of (0.1).

Let $\mathfrak{F}$ be the vector space of all complex-valued functions defined and

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real analytic on $\mathfrak{D}$ and let Q be the partial differential operator

$$Q=\Delta_2+\omega^2 \tag{0.2}$$

defined on $\mathfrak{D}$. Clearly, $Q\Phi\in\mathfrak{F}$ for $\Phi\in\mathfrak{F}$, and $\mathfrak{F}_0$ is that subspace of $\mathfrak{F}$ which is the kernel or null space of the linear operator Q.

1.1 The Symmetry Group of the Helmholtz Equation

It is a well-known fact that if $\Psi(\mathbf{x})$, $\mathbf{x}=(x,y)$, is a solution of (0.1), then $\Psi^{\wedge}(\mathbf{x})=\Psi(\mathbf{x}+\mathbf{a})$ where $\mathbf{a}=(a_1,a_2)$ is a real two-vector and $\Psi^{\wedge\wedge}(\mathbf{x})=\Psi(\mathbf{x}O)$ where

$$O(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},\qquad 0\leqslant\theta\leqslant 2\pi,$$

are also solutions. (However, $\mathbf{x}$ must be chosen so that $\mathbf{x}+\mathbf{a}$ and $\mathbf{x}O$ lie in $\mathfrak{D}$ in order for $\Psi^{\wedge}$ and $\Psi^{\wedge\wedge}$ to make sense when evaluated at $\mathbf{x}$.) Thus translations in the plane and rotations about the origin map solutions of (0.1) into solutions. These translations and rotations generate the group $E(2)$, the *Euclidean group*, whose elements are just the rigid motions in the plane. As we shall show, exploitation of this Euclidean symmetry of (0.1) yields simple proofs of many facts concerning the solutions of the Helmholtz equation. In the following paragraphs we rederive the existence of Euclidean symmetry for (0.1) and show that in a certain sense $E(2)$ is the maximal symmetry group of this equation.

We say that the linear differential operator

$$L=X(\mathbf{x})\partial_x+Y(\mathbf{x})\partial_y+Z(\mathbf{x}),\qquad X,Y,Z\in\mathfrak{F} \tag{1.1}$$

is a *symmetry operator* for the Helmholtz equation provided

$$[L,Q]=R(\mathbf{x})Q,\qquad R\in\mathfrak{F}, \tag{1.2}$$

where $[L,Q]=LQ-QL$ is the commutator of L and Q, and the analytic function $R=R_L$ may vary with L. Recall that Q is the operator (0.2). (We interpret the relation (1.2) to mean that the operators on the left- and right-hand sides yield the same result when applied to any $\Phi\in\mathfrak{F}$.)

Let $\mathfrak{G}$ be the set of all symmetry operators for the Helmholtz equation.

THEOREM 1.1. *A symmetry operator L maps solutions of* (0.1) *into solutions; that is, if $\Psi\in\mathfrak{F}_0$, then $L\Psi\in\mathfrak{F}_0$.*

Proof. If $\Psi\in\mathfrak{F}_0$, we have $\Psi\in\mathfrak{F}$ and $Q\Psi=0$. Then from (1.2), $QL\Psi=LQ\Psi-RQ\Psi=0$, so $L\Psi\in\mathfrak{F}_0$. ■

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Furthermore, it is not difficult to show that if an operator L of the form (1.1) maps solutions Ψ of $Q\Psi=0$ into solutions, then L satisfies the commutation relation (1.2) for some $R\in\mathfrak{F}$. (However, it is not known whether this statement is true for an arbitrary linear differential equation of second order.)

THEOREM 1.2. *The set $\mathcal{G}$ of symmetry operators is a complex Lie algebra; that is, if $L_1, L_2\in\mathcal{G}$, then*

(1) $a_1L_1+a_2L_2\in\mathcal{G}$ for all $a_1,a_2\in\mathcal{C}$,

(2) $[L_1,L_2]\in\mathcal{G}$.

Proof. Since $L_1,L_2\in\mathcal{G}$, these operators satisfy the equations $[L_j,Q]=R_j(\mathbf{x})Q$ where $R_j\in\mathfrak{F}$, $j=1,2$. A simple computation shows that the first-order operator $L=a_1L_1+a_2L_2$ satisfies (1.2) with $R=a_1R_1+a_2R_2$. Similarly, $L=[L_1,L_2]$ is a first-order operator that satisfies (1.2) with $R=\tilde{L}_1R_2-\tilde{L}_2R_1$ where $L=\tilde{L}+Z(\mathbf{x})$, (1.1). ■

Note: It is not excluded that $\mathcal{G}$ is an infinite-dimensional Lie algebra, although for the example considered here $\dim\mathcal{G}=4$.

We now explicitly compute the symmetry algebra of (0.1). Substituting (0.2) and (1.1) into (1.2) and evaluating the commutator, we find

$$2X_x\partial_{xx}+2(X_y+Y_x)\partial_{xy}+2Y_y\partial_{yy}+(X_{xx}+X_{yy}+2Z_x)\partial_x$$
$$+(Y_{xx}+Y_{yy}+2Z_y)\partial_y+(Z_{xx}+Z_{yy})=-R\left(\partial_{xx}+\partial_{yy}+\omega^2\right). \quad (1.3)$$

For this operator equation to be valid when applied to an arbitrary $\Phi\in\mathfrak{F}$, it is necessary and sufficient that the coefficients of ∂_{xx}, ∂_{yy}, and so on be the same on both sides of the equation:

$$\begin{aligned}&\text{(a)}\quad 2X_x=-R=2Y_y, \qquad X_y+Y_x=0,\\&\text{(b)}\quad X_{xx}+X_{yy}+2Z_x=0, \qquad Y_{xx}+Y_{yy}+2Z_y=0,\\&\text{(c)}\quad Z_{xx}+Z_{yy}=-R\omega^2.\end{aligned} \quad (1.4)$$

From equations (1.4a), $X_x=Y_y$ and $X_y=-Y_x$. Thus $X_{xx}+X_{yy}=Y_{xy}-Y_{xy}=0$; similarly, $Y_{xx}+Y_{yy}=0$. Comparing these results with equation (1.4b), we see that $Z_x=Z_y=0$, so $Z=\delta$, a constant. It follows from (1.4c) that $R=0$. Equations (1.4a) then imply $X=X(y)$, $Y=Y(x)$ with $X'(y)=-Y'(x)$. This last equation implies $X'=-Y'=\gamma\in\mathcal{C}$. Thus, the general solution of equations (1.4) is

$$X=\alpha+\gamma y, \qquad Y=\beta-\gamma x, \qquad Z=\delta, \qquad R=0, \quad (1.5)$$

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and the symmetry operator L takes the form

$$L=(\alpha+\gamma y)\partial_x+(\beta-\gamma x)\partial_y+\delta. \tag{1.6}$$

Clearly the symmetry algebra $\mathcal{G}$ is four dimensional with basis

$$P_1=\partial_x,\quad P_2=\partial_y,\quad M=y\partial_x-x\partial_y,\qquad E=1, \tag{1.7}$$

obtained by setting $\alpha=1$, $\beta=\gamma=\delta=0$ for P_1; $\beta=1$ and $\alpha=\gamma=\delta=0$ for P_2; and so on. The commutation relations for this basis are easily verified to be

$$[P_1,P_2]=0,\qquad [M,P_1]=P_2,\qquad [M,P_2]=-P_1 \tag{1.8}$$

and $[E,L]=0$ for all $L\in\mathcal{G}$. The symmetry operator E is of no interest to us, so we will ignore it and concentrate on the three-dimensional Lie algebra with basis $\{P_1,P_2,M\}$ and commutation relations (1.8). Furthermore, for reasons that will become clear shortly we will restrict our attention to the *real* Lie algebra $\mathcal{E}(2)$ generated by $\{P_1,P_2,M\}$, that is, the Lie algebra consisting of all elements $\alpha P_1+\beta P_2+\gamma M$ where α, β, γ belong to the field of real numbers R. Here, $\mathcal{E}(2)$ is isomorphic to the Lie algebra of the Euclidean group in the plane $E(2)$. To show this we consider the well-known realization of $E(2)$ as a group of 3×3 matrices. The elements of $E(2)$ are

$$g(\theta,a,b)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ a & b & 1\end{bmatrix},\qquad \begin{array}{l}a,b\in R,\\ 0\leqslant\theta<2\pi\ (\mathrm{mod}\ 2\pi),\end{array} \tag{1.9}$$

and the group product is given by matrix multiplication,

$$g(\theta,a,b)g(\theta',a',b')=g(\theta+\theta',a\cos\theta'+b\sin\theta'+a',-a\sin\theta'+b\cos\theta'+b'). \tag{1.10}$$

$E(2)$ acts as a transformation group in the plane. Indeed, the group element $g(\theta,a,b)$ maps the point $\mathbf{x}=(x,y)$ in R^2 to the point

$$\mathbf{x}g=(x\cos\theta+y\sin\theta+a,\ -x\sin\theta+y\cos\theta+b). \tag{1.11}$$

It is easy to check that $\mathbf{x}(g_1g_2)=(\mathbf{x}g_1)g_2$ for all $\mathbf{x}\in R^2$ and $g_1,g_2\in E(2)$ and that $\mathbf{x}g(0,0,0)=\mathbf{x}$ where $g(0,0,0)$ is the identity element of $E(2)$. Geometrically, g corresponds to a rotation about the origin $(0,0)$ through the angle θ in a clockwise direction, followed by the translation (a,b).

Computing the Lie algebra of the matrix group $E(2)$ in the usual way

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(see Appendix A), we find that a basis for the Lie algebra is given by the matrices

$$M=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix},\qquad P_1=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix},\qquad P_2=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\tag{1.12}$$

with commutation relations identical to (1.8). (Here, the commutator $[A,B]$ of two $n\times n$ matrices is the *matrix commutator* $[A,B]=AB-BA$.) It follows that the symmetry algebra $\mathcal{E}(2)$ is isomorphic to the Lie algebra of $E(2)$.

We can construct a general group element (1.9) from the Lie algebra elements (1.12) through use of the matrix exponential. Indeed, it is straightforward to show that

$$g(\theta,a,b)=\exp(\theta M)\exp(aP_1+bP_2)\tag{1.13}$$

where

$$\exp(A)=\sum_{k=0}^{\infty}(k!)^{-1}A^k,\qquad A^0=E_n,\tag{1.14}$$

for any $n\times n$ matrix A. Here E_n is the $n\times n$ identity matrix.

Using standard results from Lie theory (see Appendix A), we can extend the action of $\mathcal{E}(2)$ on $\mathcal{F}$ given by expressions (1.7) to a local representation $\mathbf{T}$ of $E(2)$ on $\mathcal{F}$. Indeed from Theorem A.3 we obtain the operators $\mathbf{T}(g)$ where

$$\begin{aligned}\mathbf{T}(g(0,a,0))\Phi(\mathbf{x})&=\exp(aP_1)\Phi(\mathbf{x})=\Phi(x+a,y),\\ \mathbf{T}(g(0,0,b))\Phi(\mathbf{x})&=\exp(bP_2)\Phi(\mathbf{x})=\Phi(x,y+b),\\ \mathbf{T}(g(\theta,0,0))\Phi(\mathbf{x})&=\exp(\theta M)\Phi(\mathbf{x})=\Phi(x\cos\theta+y\sin\theta,-x\sin\theta+y\cos\theta)\end{aligned}\tag{1.15}$$

and $\Phi\in\mathcal{F}$. In analogy with (1.13) the general operator $\mathbf{T}(g)$ is defined by

$$\begin{aligned}\mathbf{T}(g(\theta,a,b))\Phi(\mathbf{x})&=\exp(\theta M)\exp(aP_1)\exp(bP_2)\Phi(\mathbf{x})\\&=\Phi(\mathbf{x}g)\end{aligned}\tag{1.16}$$

where $\mathbf{x}g$ is given by (1.11). Thus the action (1.11) of $E(2)$ as a transformation group is exactly that induced by the Lie derivatives (1.7). (Recall that if L is a Lie derivative, we have by definition

$$\exp(aL)\Phi(\mathbf{x})=\sum_{k=0}^{\infty}\frac{a^k}{k!}L^k\Phi(\mathbf{x}),\qquad \Phi\in\mathcal{F};\tag{1.17}$$

see (A.8).)

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Excerpt
More information

It is a consequence of the fundamental results of Lie theory that the operators $\mathbf{T}(g)$ satisfy the group homomorphism property

$$\mathbf{T}(gg') = \mathbf{T}(g)\mathbf{T}(g'), \qquad g, g' \in E(2), \tag{1.18}$$

although the dubious reader can verify this directly. The results (1.16), (1.18) have to be interpreted with some care because for a pair $\mathbf{x} \in \mathfrak{D}$, $g \in E(2)$, the element $\mathbf{x}g$ may not lie in $\mathfrak{D}$, so that $\Phi(\mathbf{x}g)$ is undefined. However, for fixed $\mathbf{x} \in \mathfrak{D}$ the element $\mathbf{x}g$ will lie in $\mathfrak{D}$ as long as g is in a suitably small neighborhood of the identity element $g(0,0,0)$ in $E(2)$. Thus (1.16) and (1.18) have only local validity.

If L is a first-order symmetry operator of the Helmholtz equation, that is, L maps solutions into solutions, then also L^k maps solutions into solutions for each $k = 2, 3, 4, \ldots$. Furthermore, from (1.17) we see that the operator $\exp(aL)$ also maps solutions into solutions. Since the operators $\mathbf{T}(g)$ are composed of products of terms of the form $\exp(aL)$, $L \in \mathcal{E}(2)$, we can conclude that if $\Psi(\mathbf{x})$ is an analytic solution of $Q\Psi = 0$, then $\Psi'(\mathbf{x}) = \mathbf{T}(g)\Psi(\mathbf{x}) = \Psi(\mathbf{x}g)$ is also an analytic solution, with domain the open set consisting of all $\mathbf{x} \in R^2$ such that $\mathbf{x}g \in \mathfrak{D}$. (If $\mathfrak{D} = R^2$, then the operators $\mathbf{T}(g)$ are defined globally and there is no domain problem.) Based on these comments, we call $E(2)$ the *symmetry group* of the equation $Q\Psi = 0$.

It is now easy to see why we limit ourselves to the real Lie algebra with basis P_1, P_2, M. The exponential of an element of the complex Lie algebra, say iP_1 where $i = \sqrt{-1}$, is a symmetry of the Helmholtz equation. However, a straightforward application of Lie theory yields $\exp(iP_1)\Phi(\mathbf{x}) = \Phi(x + i, y)$ and this is undefined for $\Phi \in \mathfrak{F}$ because Φ is defined only for real x and y. Thus we limit ourselves to the Lie algebra whose elements have exponentials with the simple interpretation (1.16).

In analogy to our computation of the first-order symmetry operators for the Helmholtz equation, we can determine the second-order symmetry operators. We say that the second-order operator

$$S = A_{11}\partial_{xx} + A_{12}\partial_{xy} + A_{22}\partial_{yy} + B_1\partial_x + B_2\partial_y + C, \qquad A_{jk}, B_j, C \in \mathfrak{F}, \tag{1.19}$$

is a *symmetry operator* for (0.1) provided

$$[S, Q] = U(\mathbf{x})Q \tag{1.20}$$

where

$$U = H_1(\mathbf{x})\partial_x + H_2(\mathbf{x})\partial_y + J(\mathbf{x}), \qquad H_j, J \in \mathfrak{F}, \tag{1.21}$$

is a first-order differential operator. (Here U may vary with S.) We consider a first-order symmetry operator L as a special second-order symmetry. When $S = L$, equation (1.20) holds with $H_1 = H_2 = 0$. We allow

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U to be a first-order operator because the commutator of two second-order operators is an operator of order $\leqslant 3$.

The following result is proved exactly as is Theorem 1.1.

THEOREM 1.3. *A second-order symmetry operator S maps solutions of* (0.1) *into solutions; that is, if $\Psi \in \mathfrak{F}_0$, then $S\Psi \in \mathfrak{F}_0$.*

Furthermore, it is not difficult to show that if an operator S of the form (1.19) maps solutions Ψ of $Q\Psi = 0$ into solutions, then S satisfies the commutation relation (1.20) for some U of the form (1.21).

Let $\mathfrak{S}$ be the vector space of all second-order symmetry operators S. Clearly $\mathfrak{S}$ contains the first-order symmetry algebra $\mathcal{G}$. However, $\mathfrak{S}$ is not a Lie algebra under the usual bracket operation because the commutator $[S,S']$ of two second-order symmetries is in general a third-order operator, hence not an element of $\mathfrak{S}$. (Note that $[S,S']$ still maps solutions into solutions.)

Among the elements of $\mathfrak{S}$ are all operators of the form RQ where R is any element of $\mathfrak{F}$. Indeed $S = RQ$ satisfies (1.20) with $U = [R,Q]$, a first-order differential operator. We can check directly that RQ maps solutions Ψ of $Q\Psi = 0$ into other solutions. Indeed $(RQ)\Psi = R(Q\Psi) = 0$, so Ψ is mapped to the solution 0. It follows that the operators RQ are symmetries of a trivial sort; they act as the zero operator on the solution space $\mathfrak{F}_0$.

The set of all trivial symmetries $\mathfrak{q} = \{RQ : R \in \mathfrak{F}\}$ forms a subspace of $\mathfrak{S}$ and each element of $\mathfrak{q}$ acts as the zero operator on $\mathfrak{F}_0$. We will henceforth ignore $\mathfrak{q}$ and concentrate our attention on the factor space $\mathfrak{S}/\mathfrak{q}$ of nontrivial symmetries. Thus we will regard two symmetries S, S' in $\mathfrak{S}$ as identical ($S \equiv S'$) if $S' = S + RQ$ for some $R \in \mathfrak{F}$. If S is given by (1.19), then $S \equiv S'$ where $S' = S - A_{22}Q$, so that the coefficient of ∂_{yy} in the expression for S' is zero. Thus every symmetry S is equivalent to a symmetry S' whose coefficient of ∂_{yy} is zero. (Note that the operators S and S' agree on the solution space $\mathfrak{F}_0$.) Furthermore, two operators S_1, S_2 whose coefficients of ∂_{yy} are zero agree on $\mathfrak{F}_0$ if and only if their remaining coefficients are identical.

The computation of all nontrivial symmetries is straightforward. We substitute expressions (1.19) (with $A_{22} = 0$) and (1.21) into (1.20) and equate coefficients of the various partial derivatives with respect to x and y on both sides of the resulting relation. The equations obtained are analogous to (1.3) and (1.4) but somewhat more complicated. Here we present only the results of the computation.

$\mathfrak{S}/\mathfrak{q}$ is a nine-dimensional complex vector space with basis

$$\begin{aligned} &\text{(a)} \quad P_1, P_2, M, E, \\ &\text{(b)} \quad P_1^2, P_1P_2, M^2, \{M, P_1\}, \{M, P_2\}. \end{aligned} \tag{1.22}$$

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Here $\{A,B\}=AB+BA$ for operators A,B on $\mathfrak{F}$. Note that if A and B are first-order symmetries, then the products AB and BA are second-order symmetries. The results (1.22) show that the Helmholtz equation admits no nontrivial symmetries other than these; that is, all second-order symmetries are quadratic polynomials in the elements of $\mathcal{G}$. (In fact, it can be shown that the nontrivial symmetry operators of any order are polynomials in the elements of $\mathcal{G}$, but we shall not need this.) In general, if $Q\Psi=0$ is a second-order partial differential equation whose nontrivial second-order symmetries are all quadratic polynomials in the elements of the first-order symmetry algebra $\mathcal{G}$, we call such an equation *class I*. If there exists a nontrivial second-order symmetry that is not expressible as a quadratic polynomial in the first-order symmetries, the equation is called *class II*. From (1.22) we conclude that the Helmholtz equation is class I.

A few comments are in order concerning the symbol $\{\cdot,\cdot\}$. Consider the second-order symmetry MP_1. Note that

$$MP_1=\tfrac{1}{2}(MP_1+P_1M)+\tfrac{1}{2}(MP_1-P_1M)=\tfrac{1}{2}\{M,P_1\}+\tfrac{1}{2}[M,P_1].$$

Thus we have expressed MP_1 as the sum of the truly second-order (nonfirst-order) operator $\frac{1}{2}\{M,P_1\}$ and the first-order operator $\frac{1}{2}[M,P_1]=\frac{1}{2}P_2$. Similarly, any product AB of elements of $\mathcal{E}(2)$ can be written uniquely as the sum of a symmetrized purely second-order part $\frac{1}{2}\{A,B\}$ and a commutator $\frac{1}{2}[A,B]$ that belongs to $\mathcal{E}(2)$. In (1.22a) we have listed a basis for the first-order operators in $\mathcal{S}/\mathfrak{q}$, while in (1.22b) we have listed a basis for the subspace of purely second-order operators.

For another perspective on the five-dimensional space spanned by the basis (1.22b), consider the space $\mathcal{E}(2)^{(2)}$ of second-order symmetrized operators from $\mathcal{E}(2)$. This space is six-dimensional with a basis consisting of the five operators listed in (1.22b) plus the operator P_2^2. However, on $\mathfrak{F}_0$ the operator $P_1^2+P_2^2\in\mathcal{E}(2)^{(2)}$ agrees with the first-order operator $-\omega^2$, that is, multiplication by the constant $-\omega^2$. Thus to characterize those elements of $\mathcal{E}(2)^{(2)}$ which act on $\mathfrak{F}_0$ in distinct ways, we pass to the factor space $\mathcal{E}(2)^{(2)}/\{P_1^2+P_2^2\}$, where $\{P_1^2+P_2^2\}$ is the subspace of $\mathcal{E}(2)^{(2)}$ consisting of all constant multiples $a(P_1^2+P_2^2), a\in R$. This makes sense because two operators S_1, S_2 in $\mathcal{E}(2)^{(2)}$ such that $S_1-S_2=a(P_1^2+P_2^2)$ have the same eigenfunctions in $\mathfrak{F}_0$ with corresponding eigenvalues differing by $a\omega^2$.

Up to now we have considered $\mathcal{S}/\mathfrak{q}$ as the space of all *complex* linear combinations of the basis operators (1.22). However, for purposes of describing the relationship between symmetry and separation of variables for the real Helmholtz equation we shall find that it is sufficient to consider only *real* linear combinations of the basis operators (1.22). Rather than introduce a new symbol to denote this real nine-dimensional vector

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space, we shall retain the symbol $\mathcal{S}/\mathfrak{a}$ but we shall henceforth consider this vector space to be defined over R rather than $\mathcal{C}$.

With this interpretation we see that the five-dimensional subspace of purely second-order operators in $\mathcal{S}/\mathfrak{a}$ is isomorphic to $\mathcal{E}(2)^{(2)}/\{P_1^2+P_2^2\}$. That is, we can identify the purely second-order symmetries of the Helmholtz equation with the purely second-order elements in the universal enveloping algebra of $\mathcal{E}(2)$ modulo the center of the enveloping algebra. This point of view will be useful for the orbit analysis that we carry out in Section 1.2.

1.2 Separation of Variables for the Helmholtz Equation

The method of separation of variables for solving partial differential equations, although easy to illustrate for certain important examples, proves surprisingly subtle and difficult to describe in general. For this reason we begin with the simplest cases and then gradually consider cases of greater and greater complexity. At present we content ourselves with the vague assertion that separation of variables is a method for finding solutions of a second-order partial differential equation in n variables by reduction of this equation to a system of n (at most) second-order ordinary differential equations.

Let us begin by searching for solutions of (0.1) in the form $\Psi(x,y)=X(x)Y(y)$. Then the Helmholtz equation becomes

$$X''Y+XY''+\omega^2XY=0 \qquad (2.1)$$

where a prime denotes differentiation. This equation can be written

$$\frac{X''}{X}=-\frac{Y''}{Y}-\omega^2 \qquad (2.2)$$

where the left-hand side is a function of x alone and the right-hand side is a function of y alone. (Thus the Cartesian coordinates x, y have been separated in (2.2).) This is possible only if both sides of the equation are equal to a constant $-k^2$, called the *separation constant*. Thus equation (2.2) is equivalent to the pair of ordinary differential equations

$$X''(x)+k^2X(x)=0, \qquad Y''(y)+(\omega^2-k^2)Y(y)=0. \qquad (2.3)$$

A basis of solutions for the x equation is $X_1=e^{ikx}$, $X_2=e^{-ikx}$ for $k\neq 0$, while $Y_1=\exp(i(\omega^2-k^2)^{1/2}y)$, $Y_2=\exp(-i(\omega^2-k^2)^{1/2}y)$ is a basis for the y equation if $\omega^2-k^2\neq 0$. Thus we can find solutions $\Psi(x,y)$ of (0.1) in the

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form

$$\Psi_k(\mathbf{x}) = \sum_{j,l=1}^{2} A_{jl} X_j(x) Y_l(y) \tag{2.4}$$

where the complex constants A_{jl} are arbitrary. Although the Ψ_k are very special solutions of (0.1), it can be shown that essentially any solution of the Helmholtz equation can be represented as a sum or integral (with respect to k) of these special solutions.

Note that the separated solution $\Psi_k = X_1 Y_1 = \exp\{i[kx + (\omega^2 - k^2)^{1/2} y]\}$ is a simultaneous eigenvector of the commuting operators $P_1 = \partial_x$ and $P_2 = \partial_y$:

$$P_1\Psi_k = ik\Psi_k, \qquad P_2\Psi_k = i(\omega^2 - k^2)^{1/2}\Psi_k \tag{2.5}$$

with similar remarks for the other separated solutions $X_j Y_l$. Thus, we can characterize the separated solutions in Cartesian coordinates by saying that they are common eigenfunctions of the symmetry operators $P_1, P_2 \in \mathcal{E}(2)$ in $\mathcal{F}_0$.

For our next example we pass to polar coordinates r, θ:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad 0 \leqslant r, \qquad 0 \leqslant \theta < 2\pi \ (\mathrm{mod}\, 2\pi). \tag{2.6}$$

In these coordinates the Helmholtz equation becomes

$$\left(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + \omega^2\right)\Psi(r,\theta) = 0. \tag{2.7}$$

We look for solutions of the form $\Psi = R(r)\Theta(\theta)$. Substituting this expression into (2.7) and rearranging terms, we obtain

$$(r^2R'' + rR' + r^2\omega^2)R^{-1} = -\Theta''\Theta^{-1}. \tag{2.8}$$

Since the left-hand side of (2.8) is a function of r alone, while the right-hand side is a function of θ alone, both sides of this equation must be equal to a constant k^2. Thus (2.8) is equivalent to the two ordinary differential equations

$$\Theta''(\theta) + k^2\Theta(\theta) = 0, \qquad r^2R''(r) + rR'(r) + (r^2\omega^2 - k^2)R = 0. \tag{2.9}$$

The first equation has solutions $\Theta = e^{\pm ik\theta}$ while the second, Bessel's equation, admits the solutions $R = J_{\pm k}(\omega r)$ where $J_\nu(z)$ is a Bessel function (see equation (B.14)). Note that the separated solution $\Psi_k = J_k(\omega r)e^{ik\theta}$ is an eigenvector of the operator $M \in \mathcal{E}(2)$. Indeed, in polar coordinates $M =$

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