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Domenico Marinucci and Giovanni Peccati

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London Mathematical Society Lecture Notes Series: 389

# Random Fields on the Sphere

## Representation, Limit Theorems and Cosmological Applications

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Singapore, São Paulo, Delhi, Tokyo, Mexico City

Cambridge University Press  
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9780521175616](http://www.cambridge.org/9780521175616)

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First published 2011

Printed in the United Kingdom at the University Press, Cambridge

*A catalogue record for this publication is available from the British Library*

ISBN 978-0-521-17561-6 Paperback

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A Luisa, Lorenzo e Luca

A leva ed Emma Elīza

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## Preface

Several people have worked with the authors in the last years to develop the material covered in this monograph. In particular, following the order of the book, we wish to mention Ivan Nourdin and David Nualart for recent developments on the generalized method of moments for Gaussian subordinated processes, and relationships with Stein's method; Jean-Renaud Pycke for spectral representations of isotropic random fields; Paolo Baldi for the characterizations of spherical harmonic coefficients under isotropy; Paolo Baldi, Gerard Kerkycharian and Dominique Picard for the stochastic analysis of standard needlets, and Xiaohong Lan for the needlets bispectrum; Daryl Geller (who introduced Mexican needlets with Azita Mayeli) for the extension of the needlet paradigm to random sections of spin fiber bundles. We learned a lot from discussions with Mauro Piccioni and Igor Wigman, who have also provided very useful comments on an earlier draft, as did PhD students Mirko D'Ovidio and Claudio Durastanti.

The material of this book is strongly motivated by Cosmological applications, and it has benefited enormously from a decade-long interaction of the first author with physicists providing insights, suggestions, and applications to real data: we mention in particular (in alphabetic order) Amedeo Balbi, Paolo Cabella, Giancarlo de Gasperis, Frode Hansen, Michele Liguori, Sabino Matarrese, Paolo Natoli, Davide Pietrobon, Gianluca Polenta, Oystein Rudjord, Sandro Scodeller and Nicola Vittorio. Frode Hansen is to be thanked also for some insightful comments on the CMB description parts.

Our greatest thanks go to our families, to which the book is dedicated.