The three main themes of this book — probability theory, differential geometry and the theory of integrable systems — reflect the broad range of mathematical interests of Henry McKean, to whom it is dedicated.

Written by experts in probability, geometry, integrable systems, turbulence and percolation, the seventeen papers included here demonstrate a wide variety of techniques that have been developed to solve various mathematical problems in these areas. The topics are often combined in an unusual and interesting fashion to give solutions outside of the standard methods. The papers contain some exciting results and offer a guide to the contemporary literature on these subjects.

Mathematical Sciences Research Institute
Publications

55

Probability, Geometry and Integrable Systems

Mathematical Sciences Research Institute Publications

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Probability, Geometry and Integrable Systems

For Henry McKean's Seventy-Fifth Birthday

Edited by

Mark Pinsky
Björn Birnir

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Preface

This volume is dedicated to Henry McKean, on the occasion of his seventy-fifth birthday. His wide spectrum of interests within mathematics is reflected in the variety of theory and applications in these papers, discussed in the Tribute on page xv. Here we comment briefly on the papers that make up this volume, grouping them by topic. (The papers appear in the book alphabetically by first author.)

Since the early 1970s, the subject of completely integrable systems has grown beyond all expectations. The discovery that the Kortweg – de Vries equation, which governs shallow-water waves, has a complete system of integrals of motion has given rise to a search for other such evolution equations. Two of the papers in this volume, one by **Boutet de Monvel and Shepelsky** and the other by **Loubet**, deal with the completely integrable system discovered by Camassa and Holm. This equation provides a model describing the shallow-water approximation in inviscid hydrodynamics. The unknown function $u(x,t)$ refers to the horizontal fluid velocity along the x-direction at time t. The first authors show that the solution of the CH equation in the case of no breaking waves can be expressed in parametric form in terms of the solution of an associated Riemann–Hilbert problem. This analysis allows one to conclude that each solution within this class develops asymptotically into a train of solitons.

Loubet provides a technical *tour de force*, extending previous results of McKean on the Camassa–Holm equation. More specifically, he gives an explicit formula for the velocity profile in terms of its initial value, when the dynamics are defined by a Hamiltonian that is the sum of the squares of the reciprocals of a pair of eigenvalues of an associated acoustic equation. The proof depends on the analysis of a simpler system, whose Hamiltonian is defined by the reciprocal of a *single* eigenvalue of the acoustic equation. This tool can also solve more complex dynamics, associated to several eigenvalues, which eventually leads to a new proof of McKean's formula for the Fredholm determinant. The paper concludes with an asymptotic analysis (in both past and future directions), which allows partial confirmation of statements about soliton genesis and interaction that were raised in an earlier CPAM paper.

Meanwhile we have a contribution from **Gibbons, Holm and Tronci** of a more geometric nature. This deals with the Vlaslov equation, which describes the evolution of the single-particle probability density in the evolution of N point particles. Specifically, they study the evolution of the p-th moments when the dynamics is governed by a quadratic Hamiltonian. The resulting motion takes place on the manifold of symplectomorphisms, which are smooth invertible maps acting on the phase space. The singular solutions turn out to be closely related to integrable systems governing shallow-water wave theory. In fact, when these equations are "closed" at level p, one retrieves the peaked solitons of the integrable Camassa–Holm equation for shallow-water waves!

Segur's paper provides an excellent overview of the development of our understanding of integrable partial differential equations, from the Boussinesq equation (1871) to the Camassa–Holm equation (1993) and their relations to shallow-water wave theory. In physical terms, an integrable system is equivalent to the existence of *action-angle variables*, where the *action variables* are the integrals of motion and the angle variables evolve according to simple ordinary differential equations. Since each of these PDEs also describes waves in shallow water, it is natural to ask the question: *Does the extra mathematical structure of complete integrability provide useful information about the behavior of actual physical waves in shallow water?* The body of the paper takes up this question in detail with many illustrations of real cases, including the tsunami of December 26, 2004. A video link is provided, for further documentation.

Previato's paper contains a lucid account of the use of theta functions to characterize lines in abelian varieties. Strictly speaking, "line" is short for linear flow, since an abelian variety cannot properly contain a (projective) line. More than twenty years ago Barsotti had proved that, on any abelian variety, there exists a direction such that the derivatives of sufficiently high order of the logarithim of the theta function generate the function field of the abelian variety. The purpose of the present paper is to use the resulting differential equations to characterize theta functions, a generalization of the KP equations, introduced by Kadomtsev and Petviashvili in 1971, as well as to study spectra of commutative rings of partial differential operators.

Arov and **Dym** summarize their recent work on inverse problems for matrix-valued ordinary differential equations. This is related to the notion of reproducing kernel Hilbert spaces and the theory of J-inner matrix functions. A time-independent Schrödinger equation is written as a system of first-order equations, which permits application of the basic results.

Cruzeiro and **Malliavin** study a first-order Burgers equation in the context of flows on the group of diffeomorphisms of the circle, an infinite-dimensional Riemannian manifold. The L^2 norm on the circle defines the Riemannian metric, so that the Burgers equation defines a geodesic flow by means of an ordinary differential equation which flows along the Burgers trajectories. The authors compute the connection coefficients for both the Riemannian connection defined by the parallel transport and the algebraic connection defined by the right-invariant parallelism. These computations are then used to solve a number of problems involving stochastic parallel transport, symmetries of the noise, and control of ultraviolet divergence with the help of an associated Markov jump process.

The paper of **Cambronero**, **Ramírez** and **Rider** describes various links between the spectrum of random Schrödinger operators with particular emphasis on Hill's equation and random matrix theory. The unifying theme is the utility of the Riccati map in converting problems about second-order differential operators and their matrix analogues into questions about one-dimensional diffusions. The paper relies on functional integration to derive many interesting results on the spectral properties of random operators. The paper provides an excellent overview of results on Hill's equation, including the exploitation of low-lying eigenvalues. The penultimate section of the paper describes the recent and exciting developments involving Tracy–Widom distributions and their far-reaching generalizations to all positive values of the inverse temperature β.

Birnir's paper provides an excellent overview of his recent results on unidirectional flows. This is a special chapter in the theory of turbulence, and is not commonly presented. This type of modeling is important in the study of fluvial sedimentation that gives rise to sedimentary rock in petroleum reservoirs. The flow properties of the rock depend strongly on the topological structure of the meandering river channels. The methods developed can also be applied to problems of atmospheric turbulence. Contrary to popular belief, the turbulent temperature variations in the atmosphere may be highly anisotropic, nearly stratified. Thus, the scaling first developed in the case of a river or channel may have a close analogue in the turbulent atmosphere.

Halfway between probability theory and classical physics is the subject of statistical mechanics. In the paper of **Costeniuc, Ellis, Touchette** and **Turkington**, the Gaussian ensemble is introduced, to complement the micro-canonical and macro-canonical ensembles that have been known since the time of Gibbs. It is demonstrated that many minimization problems in statistical physics are most effectively expressed in terms of the Gaussian ensemble.

Grinevich and **Novikov** present a lucid overview of their work on finding formulas for the topological charge and other quantities associated with the sine-Gordon equation, which describes immersions of negatively curved surfaces into $\mathbb{R}^3$. Non-singular real periodic finite-gap solutions of the sine-Gordon equation are characterized by a genus g hyperelliptic curve whose branch points are either real positive or form complex conjugate pairs. The authors describe the admissible branch points as zeros of a meromorphic differential of the third kind, which in turn is defined by a real polynomial $P(\lambda)$ of degree $g-1$. This leads to a formula for the topological charge of these solutions, which was first given in a 1982 paper of Dubrovin and Novikov. The proof relates the topological charge to a set of certain integer characteristics of the polynomial $P(\lambda)$. The methods developed here can be applied, with suitable modifications, to the KdV equation, the defocusing nonlinear Schrödinger equation and the Kadomtsev–Petvishvilii equation.

The paper of **Ercolani** and **McLaughlin** investigates a system of equations which originate in the physics of two-dimensional quantum gravity, the so-called loop equations of random matrix theory. The analysis depends on an asymptotic formula, of the large deviation type, for the partition function in the Unitary Ensemble of random matrix theory. The loop equations are satisfied by the coefficients in a Laurent expansion expressing certain Cauchy-like transforms in terms of a quadratic expression in the derivatives. The final paragraph of the paper suggests some open problems, such as the following: to use the loop equations to find closed form expressions for the expansion coefficients of the logarithm of the partition function when the dimension $N \to \infty$; it is also anticipated that the loop equations could be used to determine qualitative behavior of these coefficients.

Manna and **Moll** offer a beautiful set of generalizations of the classical Landen transform, which states that a certain elliptic integral of the first kind containing two parameters, when expressed in trigonometric form, is invariant under the transformation defined by replacing the parameters by their arithmetic (resp. geometric) means. Later Gauss used this to prove that the limit of the iterates of this transformation exist and converge to the reciprocal of this elliptic integral, suitably modified. This limit is, by definition, the *arithmetic-geometric mean* of the initial conditions. This idea is generalized in several directions, the first of which is to a five-parameter set of rational integrals, where the numerator is of order four and the denominator of order six. The requisite Landen transformation has a simple geometric interpretation in terms of doubling the angle of the cotangent function in the trigonometric form of the integrand. The remainder of the paper describes generalizations to higher-order rational integrands, where the doubling of the cotangent function is replaced by a magnification of order $m \geq 2$. It is proved that the limit of the Landen transforms exists and can be represented as a suitable integral. All of the models studied in this paper can be considered as discrete-time (partially) integrable dynamical systems where the conserved quantity is the definite integral that is invariant under the change of parameters. The simplest of these is the elliptic integral studied by Landen, where the dynamics is defined by the arithmetic-geometric mean substitution.

Varadhan has contributed a lucid overview of his recent joint work on homogenization. In general, this theory leads to approximations of solutions of a differential equation with rapidly varying coefficients in terms of solutions of a closely related equation with constant coefficients. A model problem is the second-order parabolic equation in one space dimension, where the constant coefficient is the harmonic mean of the given variable coefficient equation. This analytical result can be expressed as a limit theorem in probability, specifically an ergodic theorem for the variable coefficient, when composed with the diffusion process defined by the parabolic equation; the normalized invariant measure is expressed in terms of the harmonic mean. Having moved into the probabilistic realm, one may just as well consider a second-order parabolic equation with random coefficients, where the results are a small variation of those obtained in the non-random case. With this intuitive background, it is natural to expect similar results when one begins with a d-dimensional equation of Hamilton–Jacobi type, defined by a convex function and with a small noise term. In joint work with Kosygina and Rezhakanlou, it is proved that the noise disappears and the HJ solution converges to the solution of a well-determined first-order equation, where the homogenized convex function is determined by a convex duality relation. The reader is invited to pursue the details, which are somewhat parallel to the model case described above.

Camia and **Newman**'s paper relies on the stochastic Schramm–Loewner equation (SLE), which provides a new and powerful tool to study scaling limits of critical lattice models. These ideas have stimulated further progress in understanding the conformally invariant nature of the scaling limits of several such models. The paper reviews some of the recent progress on the scaling limit of two-dimensional critical percolation, in particular the convergence of the exploration path to chordal SLE and the "full" scaling limit of cluster interface loops. The results on the full scaling limit and its conformal invariance are presented here for the first time. For site percolation on the triangular lattice, the results are fully rigorous and the main ideas are explained.

Grünbaum's paper proposes a new spectral theory for a class of discrete-parameter Markov chains, beginning with the case of the birth-death process, studied by Karlin/McGregor in the 1950s. More generally, for each Markov chain there is a system of orthogonal polynomials which define the spectral decomposition. Explicit computation of the relevant orthogonal polynomials is available for other Markov chains, such as random walk on the N-th roots of unity and the processes associated with the names of Ehrenfest and Tchebychev.

A principal emphasis here, due originally to M. G. Krein, is the formulation of *matrix-valued* orthogonal polynomials. Several solved examples are presented, while several natural open problems are suggested for the adventurous reader.

Van Moerbeke provides a masterful account of the close connection between nonintersecting Brownian motions and integrable systems, where the connection is made in terms of the theory of orthogonal polynomials, using previous results of Adler and Van Moerbeke. If N Brownian particles are started at p definite points and required to terminate at q other points in unit time, the object is to study the distribution as $N \to \infty$, especially in the short time limit $t \downarrow 0$ and the unit time limit $t \uparrow 1$. When the supports of these measures merge together, we have a *Markov cloud*, defined as an infinite-dimensional diffusion process depending only on the nature of the various possible singularities of the equilibrium measure. The connection between non-intersecting Brownian motion and orthogonal polynomials begins with a formula of Karlin and McGregor which expresses the non-intersection probability at a fixed time as the N-dimensional integral of the product of two determinants. Numerous special cases are provided to illustrate the general theory.

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We also thank Professor Anne Boutet de Monvel for making available to us information documenting McKean's honorary doctorate from the Université Paris VII in May 2002.

Mark A. Pinsky
Department of Mathematics
Northwestern University
Evanston, IL 60208-2370, USA

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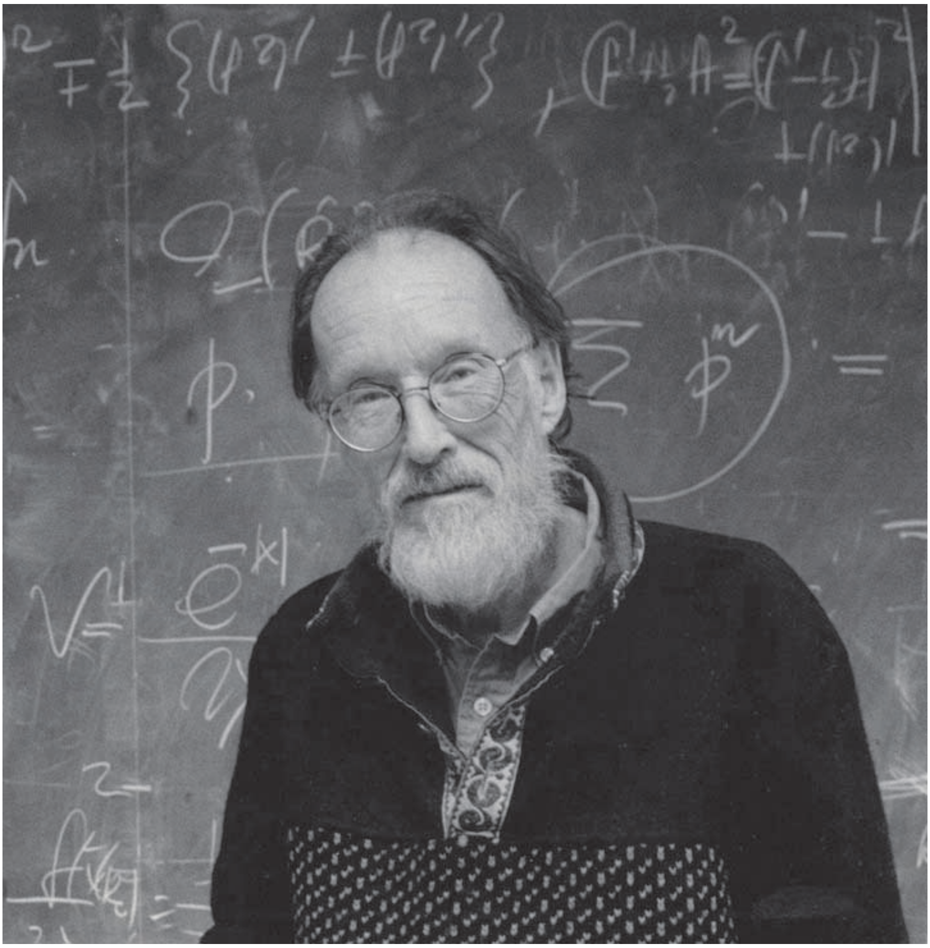
Henry McKean

A TRIBUTE BY THE EDITORS



The significance of McKean's work

Henry McKean has championed a unique viewpoint in mathematics, with good taste and constant care toward a balance between the abstract and the concrete. His great influence has been felt through both his publications and his teaching. His books, all of which have become influential, can be recognized by their concise, elegant and efficient style.

His interests include probability theory, stochastic processes, Brownian motion, stochastic integrals, geometry and analysis of partial differential equations, with emphasis on integrable systems and algebraic curves, theta functions, Hill's equation and nonlinear equations of the KdV type. He also was a pioneer in the area of financial mathematics before it became a household word.

The importance of stochastic models in modern applied mathematics and science cannot be overestimated. This is well documented by the large number of diverse papers in this volume that are formulated in a stochastic context. Henry McKean was one of the early workers in the theory of diffusion processes, as documented in his classic work with K. Itô, *Diffusion Processes and Their Sample Paths* (Springer, 1965). This was followed by his *Stochastic Integrals* (Academic Press, 1969). This book led the way to understanding the close connections between probability and partial differential equations, especially in a geometric setting (Lie groups, Riemannian manifolds).

On January 6, 2007, McKean was awarded the Leroy P. Steele Prize for Lifetime Achievement, presented annually by the American Mathematical Society. The prize citation honors McKean for his "rich and magnificent mathematical career" and for his work in analysis, which has a strong orientation towards probability theory; it states further that "McKean has had a profound influence on his own and succeeding generations of mathematicians. In addition to the important publications resulting from his collaboration with Itô, McKean has written several books that are simultaneously erudite and gems of mathematical exposition. As his long list of students attest, he has also had enormous impact on the careers of people who have been fortunate enough to study under his direction."

McKean's published work includes five books and more than 120 articles, in such journals as the *Annals of Mathematics*, *Acta Mathematica* and *Inventiones Mathematicae*, to name a few. To illustrate the richness of the mathematics he has been involved with, we take a brief look at published reviews of his books and then discuss the main threads of his research articles by subject.

1. Diffusion Processes and Their Sample Paths (with Kiyosi Itô). *Grundlehren der mathematischen Wissenschaften* **125**, Springer, New York, 1965; second (corrected) printing, 1974; reprinted in *Classics in Mathematics*, 1996.

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We adapt the comments by T. Watanabe in *Mathematical Reviews*: Feller's work on linear diffusion was primarily of an analytic character. This spurred some outstanding probabilists (including the authors) to reestablish Feller's results by probabilistic methods, solving some conjectures of Feller and studying profoundly the sample paths of one-dimensional diffusion. Their purpose is to extend the theory of linear diffusion to the same level of understanding which Paul Lévy established for Brownian motion. This is completely realized in this book by combining special tools such as Brownian local time with the general theory of Markov processes. This book is the culmination of a ten-year project to obtain the general linear diffusion from standard Brownian motion by time change and killing involving local times.

On the occasion of the book's republishing in Springer's *Classics in Mathematics* series, the cover blurb could boast without the least exaggeration: "Since its first publication … this book has had a profound and enduring influence on research into the stochastic processes associated with diffusion phenomena. Generations of mathematicians have appreciated the clarity of the descriptions given of one or more dimensional diffusion processes and the mathematical insight provided into Brownian motion."

2. *Stochastic Integrals.* Academic Press, New York, 1969.

From comments by E. B. Dynkin in *Mathematical Reviews:* "This little book is a brilliant introduction to an important interface between the theory of probability and that of differential equations. The same subject was treated in the recent book of I. I. Gihman and A. V. Skorokhod. The author's book is smaller, contains more examples and applications and is therefore much better suited for beginners. Chapter 1 is devoted to Brownian motion. Chapter 2 deals with stochastic integrals and differentials. Chapter 3 deals with one-dimensional stochastic integral equations. In Chapter 4, stochastic integral equations on smooth manifolds are investigated. Winding properties of planar Brownian motion (about one or two punctures) are deduced from the study of Brownian motion on Riemann surfaces. The last three sections are devoted to constructing Brownian motion on a Lie group, starting from Brownian motion on the Lie algebra. by means of the so-called product integral and the Maljutov–Dynkin results about Brownian motion with oblique reflection. In treating the applications of stochastic integrals, the author frequently explains the main ideas by means of typical examples, thus avoiding exhausting generalities. This remarkable book will be interesting and useful to physicists and engineers (especially in the field of optimal control) and to experts in stochastic processes."

3. ***Fourier Series and Integrals*** (with Harry Dym). Academic Press, New York, 1972.

Elliot Lieb of Princeton University wrote: "In my opinion, the book of Dym and McKean is unique. It is a book on analysis at an intermediate level with a focus on Fourier series and integrals. The reason the book is unique is that books on Fourier analysis tend to be quite abstract, or else they are applied mathematics books which give very little consideration to theory. This book is a solid mathematics book but written with great fluency and many examples. In addition to the above considerations, there is also the fact that the literary style of the authors is excellent, so that the book has a readability that is rarely found in mathematics books, especially in modern texts."

4. ***Stationary Gaussian Processes*** (with Harry Dym). Academic Press, New York, 1976.

From comments by S. Kotani in *Mathematical Reviews:* "This is a monograph on stationary Gaussian processes in one dimension. Given a mean zero Gaussian process $x(t)$, $t \in \mathbb{R}$, one may ask the following questions: (i) to predict the future given the past $-\infty < t < 0$; (ii) to predict the future given the finite segment of the past $-2T < t < 0$; (iii) to predict $x(t)$ for $|t| < T$ given $x(t)$ for $|t| > T$; (iv) the degree of dependence of the future on the past; (v) the degree of mixing of the process $x(t)$. This book contains a clear and concise introduction to the subject, often original presentations of known results in addition to several new results. The solution to problem (i) goes back to Kolmogorov, while the solution of (iii) is due to M. G. Krein in 1954. The authors re-work the solution of Krein and from that point they completely solve problem (ii). They also make important contributions to the understanding of problems (iv) and (v)."

5. ***Elliptic Curves: Function Theory, Geometry and Arithmetic*** (with Victor Moll). Cambridge University Press, Cambridge, 1997.

From W. Kleinert's review in *Zentralblatt für Mathematik*: "Altogether, this highly non-standard textbook provides the reader with ... a deep insight into historically known mathematical interrelations and references to modern developments in the analysis, geometry and arithmetic of elliptic curves. The book reflects the authors' profound knowledge and deep devotion to the historical development of the theory of elliptic curves. In these days it is certainly very profitable for the mathematical community to have such a book among the increasing number of others on the subject ... this book is not only a perfect primer for beginners in the field, but also an excellent source for researchers in various areas of mathematics and physics."

Peter Sarnak agrees: "Very unusual in covering the important aspects of elliptic curves (analytic, geometric and arithmetic) and their applications — in a single reasonably sized volume. This account of the subject, in the style of the original discoverers is, in my opinion, the best way to present the material in an introductory book."

Works by subject

Given his wide spectrum of interest, it is difficult to summarize McKean's contributions in a few paragraphs. We mention some of the principal themes:

Questions in the theory of probability and stochastic processes, beginning with Brownian motion and leading to the completion of W. Feller's program, to found the theory of one-dimensional diffusion on a probabilistic basis rather than the analytical foundation (Hille–Yosida theorem) that was standard before the 1950s. One probabilistic approach is to use the natural scale and speed measure to obtain the diffusion process directly from Brownian motion. This approach is well documented in Itô–McKean. The other approach is to develop Itô's theory of stochastic differential equations (SDE) to obtain the diffusion as the image of a Brownian motion process via a nonlinear mapping that defines the solution operator of the SDE. This approach, which works equally well in higher dimensions, is well documented in his *Stochastic Integrals*.

Geometry and analysis of partial differential equations, especially integrable systems and algebraic curves, theta functions and Hill's equation. Hill's equation is the one-dimensional Schrödinger equation

$$-y'' + q(x)y = \lambda y$$

where the potential function $q(x)$ is periodic. The spectrum of the operator $L = -d^2/dx^2 + q$ associated to such an equation is generally made up of an infinite number of intervals. But it can happen that, in limiting cases, the spectrum is formed of a finite number of n intervals and an infinite interval $\lambda > \lambda_{2n}$. For which potentials does this occur? Several authors have shown that this phenomenon requires that the potential be a solution of some auxiliary differential equations. Then Peter Lax showed in 1972/1974 the relation with certain solutions of the KdV equation. McKean and van Moerbeke solved the problem by establishing a close relation with the classic theory of hyperelliptic functions. Then McKean and Trubowitz in 1976/1978 extended the results to the case $n = \infty$, showing that the periodic spectrum of the Hill operator is infinite. This work proved to be the starting point of a series of new developments associated with the names of McKean and Trubowitz, Feldman, Knörrer, Krichever and Merkl.

Geometry of KdV equations. The Korteweg–de Vries equation

$$\frac{\partial q}{\partial t} = 3q\,\frac{\partial q}{\partial x} - \frac{1}{2}\frac{\partial^3 q}{\partial x^3}$$

describes the propagation of a wave $q(x,t)$ in a shallow canal. It has certain common features with other nonlinear partial differential equations: KdV has the structure of the equations of Hamiltonian dynamics; it has certain "solitary solutions" rather than wave train solutions; finally, it has a rich set of constants of motion related to the spectra of certain associated equations. These equations appear as "isospectral deformations" of some natural operators. This series of articles, devoted to the geometry of KdV, is based on tools from algebraic geometry (in dimension ∞), especially hyperelliptic curves, their Jacobians, theta functions and their connections with the spectral theory of Hill's equation.

Nonlinear equations. The nonlinear equations that interested McKean are the classical commutation relations

$$\frac{\partial L}{\partial t} = [L, K],$$

where L is a differential operator of first order whose coefficients are 2×2 matrices of class C^∞ and where K is an antisymmetric operator of the same type. The KdV equation

$$\frac{\partial q}{\partial t} = 3q\,\frac{\partial q}{\partial x} - \frac{1}{2}\frac{\partial^3 q}{\partial x^3}$$

is of this type where L is the Hill operator

$$\begin{aligned} L &:= -\Delta + q, \\ K &:= 2D^3 - \tfrac{3}{2}(qD + Dq), \end{aligned}$$

where $D = \partial/\partial x$ and $\Delta = D^2$. In addition to these equations, McKean studied the sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2} - \sin q$$

and the equations of Boussinesq and Camassa–Holm, written, respectively, as

$$\frac{\partial^2 q}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left(\frac{4}{3}q^2 + \frac{1}{3}\frac{\partial^2 q}{\partial x^2}\right),$$

$$\frac{\partial u}{\partial t} + 3u\frac{\partial u}{\partial x} = \alpha^2\left(\frac{\partial^3 u}{\partial x^2 \partial t} + 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^4 u}{\partial x^4}\right).$$

Financial mathematics. In 1965, following a suggestion of the economist Paul Samuelson, McKean wrote a short article on the problem of “American options”. This was first published as an appendix to a treatise by Samuelson on financial economics. It anticipated by eight years the famous Black–Scholes–Merton formula. In this paper, McKean shows that the price of an American option can be computed by solving a free boundary problem for a parabolic partial differential equation. This was the first application of nonlinear partial differential equations in financial mathematics.

Some personal tributes

Paul Malliavin: In 1972 I was working in the theory of functions of several complex variables, more specifically on the characterization of the set of zeros of a function of the Nevanlinna class in a strictly pseudo-convex domain of $\mathbb{C}^n$. In the special case of the ball, by using semisimple harmonic analysis, I computed exactly the Green function and obtained the desired characterization. For the general case I was completely stuck; I looked at the canonical heat equation associated to an exhaustion function; a constructive tool to evaluate the Green function near the boundary was urgently needed. Then I started to study Itô’s papers on SDEs; their constructive essence seemed to me quite appropriate for this estimation problem.

Then McKean’s *Stochastic Integrals* appeared: I was fully rescued in my efforts to grasp from scratch the theory of SDEs. McKean’s book is short, with carefully written concrete estimates; it presents with sparkling clarity a conceptual vision of the theory. In a note in the *Proceedings of the National Academy* I found the needed estimate for the Green function. From there I started to study the case of weakly pseudoconvex domains, where hypoelliptic operators of Hörmander type appeared; from that time onward (for the last thirty years) I became fully involved in probability theory. So Henry has had a key influence on this turning point of my scientific interests.

From the middle seventies Henry has kindly followed the different steps of my career, supporting me at every occasion. For instance, when I was in the process of being fired from Université Paris VI in 1995, he did not hesitate to cross the Atlantic in order to sit on a special committee, specially constituted by the president of Université Paris VI, in order to judge if my current work was then so obsolete that all the grants for my research associates had to be immediately eliminated.

With gratitude I dedicate our paper to Henry; also with admiration, discovering every day more and more the breadth and the depth of his scientific impact.

Daniel Stroock: I have known Henry since 1964, the year he visited Rockefeller Institute and decided to leave MIT and New England for the city of New York.

I do not know any details of Henry's childhood, but I have a few impressions which I believe bear some reasonable resemblance to the truth. Henry was the youngest child of an old New England family and its only son. He grew up in the North Shore town of Beverly, Massachusetts, where he developed a lasting love for the land, its inhabitants and the way they pronounced the English language. I gather that Henry was younger than his siblings, so he learned to fend for himself from an early age, except that he was taught to ski by his eldest sister, who was a superb athlete. He was sent to St. Paul's for high school where, so far as I know, his greatest distinction was getting himself kicked out for smoking. Nonetheless, he was accepted to Dartmouth College, where, under the influence of Bruce Knight, he began to develop a taste for mathematics. His interests in mathematics were further developed by a course which he took one summer from Mark Kac, who remained a lasting influence on Henry.

In fact, it was Kac who invited Henry to visit and then, in 1966, move to Rockefeller Institute. Since I finished my PhD the spring before Henry joined the faculty, I do not know many details of life in the Rockefeller mathematics-physics group during the late 1960s. However I do know one story from that period which, even if it's not totally true, nicely portrays Henry's affection for Kac, the founding father of the Rockefeller Mathematics Department. As such, it was his job to make it grow. For various reasons, not the least of which was his own frequent absence, Kac was having limited success in this enterprise; at one faculty meeting Kac solicited suggestions from his younger colleagues. Henry's suggestion was that they double Kac's salary in order to have him there on a full-time basis.

Henry's mathematical achievements may be familiar to anyone who is likely to be reading this book, with one proviso: most mathematicians do not delve into the variety of topics to which Henry has contributed. Aside from the constant evidence of his formidable skills, the property shared by all of Henry's mathematics is a strong sense of *taste*. Whether it is his early collaboration with Itô, his excursion into Gaussian prediction theory, or his interest in completely integrable systems and spectral invariance, Henry has chosen problems because they interest him and please his sense of aesthetics. As a result, his mathematics possesses originality all of its own, and a beauty that the rest of us can appreciate.

Srinivasa Varadhan, former director of the Courant Insitute: Henry has been my colleague for nearly thirty-five years. I have always been impressed by the number of students Henry has produced. I checked the Genealogy project. It lists him with nearly fifty students. It is even more impressive that he has nearly three hundred descendants, which means he has taught his students how to teach.

Henry is known for meticulous attention to detail. When Dan Stroock and I wrote our first paper on the martingale problem we gave the manuscript to Henry for comments. The paper was typewritten, before the days of word processors and xerox machines. Henry gave it back to us within days and the comments filled out all the empty space between the lines on every page. The typist was not amused.

Henry's own work drifted in the seventies out of probability theory into integrable systems and then back again at some point into probability. Talking to Henry about any aspect of mathematics is always fun. He has many interesting but unsolvable problems and will happily share them with you. If I come across a cute proof of something Henry is the first one I will think of telling.

Charles Newman, member and former Director of the Courant Institute: Henry's earliest papers (*Ann. Math.* 1955) concerned sample functions of stable processes, together with several papers on Brownian local time (*J. Math. Kyoto* 1962, *Adv. Math.* 1975). These topics and the general subject of one-dimensional diffusions were of course analyzed in great detail in Henry's celebrated 1965 book with Itô. I mention these all because they play a major role in my own much later work (jointly with Fontes and Isopi, *Ann. Prob.* 2002) on scaling limits of random walks with random rates to singular diffusions (with random speed measures).

There are a number of other connections between papers of Henry's and my own — typically with a multidecade gap. For example, there are close connections between Henry's paper "Geometry of differential space" (*Ann. Prob.* 1973) and my 2003 paper with D'Aristotile and Diaconis, "Brownian motion and the classical groups".

The paper by Camia and myself in this volume is less directly related to Henry's work. However, the general subject of Schramm–Loewner evolutions, to which our paper belongs, combines many of the same themes that have permeated Henry's work: Brownian motions and related processes, complex variable theory, and statistical mechanics.