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Bitangential direct and inverse problems for systems of differential equations

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ABSTRACT. A number of results obtained by the authors on direct and inverse problems for canonical systems of differential equations, and their implications for certain classes of systems of Schrödinger equations and systems with potential are surveyed. Connections with the theory of J-inner matrix valued and reproducing kernel Hilbert spaces, which play a basic role in the original developments, are discussed.

1. Introduction

In this paper we shall present a brief survey of a number of results on direct and inverse problems for canonical integral and differential systems that have been obtained by the authors over the past several years. We shall not attempt to survey the literature, which is vast, or to compare the methods surveyed here with other approaches. The references in [Arov and Dym 2004; 2005b; 2005c] (the last of which is a survey article) may serve at least as a starting point for those who wish to explore the literature.

The differential systems under consideration are of the form

$$y'(t,\lambda) = i\lambda y(t,\lambda)H(t)J, \quad 0 \leqslant t < d\,, \tag{1.1}$$

where $H(t)$ is an $m \times m$ locally summable mvf (matrix valued function) that is positive semidefinite a.e. on the interval $[0,d)$, J is an $m \times m$ signature matrix, i.e., $J = J^*$ and $J^*J = I_m$, and $y(t,\lambda)$ is a $k \times m$ mvf.

The matrizant or fundamental solution, $Y_t(\lambda) = Y(t,\lambda)$, of (1.1) is the unique locally absolutely continuous $m \times m$ solution of (1.1) that meets the initial condition $Y_0(\lambda) = I_m$, i.e.,

$$Y_t(\lambda) = I_m + i\lambda \int_0^t Y_s(\lambda)H(s)\,ds\,J \quad \text{for } 0 \le t < d\,. \tag{1.2}$$

Standard estimates yield the following properties:

(1) $Y_t(\lambda)$ is an entire mvf that is of exponential type in the variable λ for each fixed $t \in [0, d)$.

(2) The identity

$$\frac{J - Y_t(\lambda) J Y_t(\omega)^*}{-2\pi i(\lambda - \overline{\omega})} = \frac{1}{2\pi} \int_0^t Y_s(\lambda) H(s) Y_s(\omega)^* ds \tag{1.3}$$

holds for each $t \in [0, d)$ and for every pair of points $\lambda, \omega \in \mathbb{C}$.

(3) $Y_t(\lambda)$ is J-inner (in the variable λ) in the open upper half plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \lambda + \overline{\lambda} > 0\}$ for each fixed $t \in [0, d)$. This means that

$$J - Y_t(\omega) J Y_t(\omega)^* \geq 0 \quad \text{for } \omega \in \mathbb{C}_+ ,$$

with equality if $\omega \in \mathbb{R}$.

(4) $J - Y_t(\omega) J Y_t(\overline{\omega})^* = 0 \quad \text{for } \omega \in \mathbb{C}$.

(5) The kernel

$$K_\omega^t(\lambda) = \frac{J - Y_t(\lambda) J Y_t(\omega)^*}{-2\pi i(\lambda - \overline{\omega})}$$

is positive in the sense that

$$\sum_{i,j=1}^n u_i^* K_{\omega_j}^t(\omega_i) u_j \geqslant 0$$

for every choice of the points $\omega_1, \ldots, \omega_n$, vectors $u_1, \ldots, u_n$ and every positive integer n.

(6) $Y_{t_1}^{-1} Y_{t_2}$ is also an entire J-inner mvf for $0 \leq t_1 \leq t_2 < d$.

Every $m \times m$ signature matrix $J \neq \pm I_m$ is unitarily equivalent to the matrix

$$j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad p + q = m,$$

with

$$p = \operatorname{rank}(I_m + J) \geq 1 \quad \text{and} \quad q = \operatorname{rank}(I_m - J) \geq 1 .$$

The main examples of signature matrices, apart from $\pm j_{pq}$, are $\pm j_p$, $\pm J_p$ and $\pm \mathcal{J}_p$, where

$$J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix}, \quad j_p = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad \mathcal{J}_p = \begin{bmatrix} 0 & -iI_p \\ iI_p & 0 \end{bmatrix}.$$

In the last three examples $q = p$ (so that $2p = m$). The signature matrices J_p and j_p are connected by the signature matrix

$$\mathfrak{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} -I_p & I_p \\ I_p & I_p \end{bmatrix}, \quad \text{i.e.,} \quad \mathfrak{V} J_p \mathfrak{V} = j_p \quad \text{and} \quad \mathfrak{V} j_p \mathfrak{V} = J_p \,.$$

There is a natural link between each of the three principal signature matrices and each of the following three classes of mvf's that are holomorphic in $\mathbb{C}_+$, the open upper half plane:

(1) The Schur class $\mathcal{S}^{p\times q}$ of $p\times q$ mvf's $\varepsilon(\lambda)$ that are holomorphic in $\mathbb{C}_+$ and satisfy the constraint $I_q - \varepsilon(\lambda)^*\varepsilon(\lambda) \geq 0$, since

$$I_q - \varepsilon(\lambda)^*\varepsilon(\lambda) \geq 0 \Longleftrightarrow [\,\varepsilon(\lambda)^* \quad I_q\,]\, j_{pq} \begin{bmatrix} \varepsilon(\lambda) \\ I_q \end{bmatrix} \leq 0\,. \tag{1.4}$$

(2) The Carathéodory class $\mathcal{C}^{p\times p}$ of $p\times p$ mvf's $\tau(\lambda)$ that are holomorphic in $\mathbb{C}_+$ and satisfy the constraint $\tau(\lambda) + \tau(\lambda)^* \geq 0$, since

$$\tau(\lambda) + \tau(\lambda)^* \geq 0 \Longleftrightarrow [\,\tau(\lambda)^* \quad I_p\,]\, J_p \begin{bmatrix} \tau(\lambda) \\ I_p \end{bmatrix} \leq 0\,. \tag{1.5}$$

(3) The Nevanlinna class $\mathcal{R}^{p\times p}$ of $p\times p$ mvf's $\tau(\lambda)$ that are holomorphic in $\mathbb{C}_+$ and satisfy the constraint $(\tau(\lambda) - \tau(\lambda)^*)/i \geq 0$, since

$$(\tau(\lambda) - \tau(\lambda)^*)/i \geq 0 \Longleftrightarrow [\,\tau(\lambda)^* \quad I_p\,]\, \mathcal{J}_p \begin{bmatrix} \tau(\lambda) \\ I_p \end{bmatrix} \leq 0\,, \tag{1.6}$$

A general $m\times m$ mvf $U(\lambda)$ is said to be J-inner with respect to the open upper half plane $\mathbb{C}_+$ if it is meromorphic in $\mathbb{C}_+$ and if

(1) $J - U(\lambda)^* J U(\lambda) \geq 0$ for every point $\lambda \in \mathfrak{h}_U^+$ and
(2) $J - U(\mu)^* J U(\mu) = 0$ a.e. on $\mathbb{R}$,

in which $\mathfrak{h}_U^+$ denotes the set of points in $\mathbb{C}_+$ at which U is holomorphic. This definition is meaningful because every mvf $U(\lambda)$ that is meromorphic in $\mathbb{C}_+$ and satisfies the first constraint automatically has nontangential boundary values. The second condition guarantees that $\det U(\lambda) \not\equiv 0$ in $\mathfrak{h}_U^+$ and hence permits a pseudo-continuation of $U(\lambda)$ to the open lower half plane $\mathbb{C}_-$ by the symmetry principle

$$U(\lambda) = J\{U^\#(\lambda)\}^{-1} J \quad \text{for } \lambda \in \mathbb{C}_-\,,$$

where $f^\#(\lambda) = f(\overline{\lambda})^*$.

The symbol $\mathcal{U}(J)$ will denote the class of J-inner mvf's considered on the set $\mathfrak{h}_U$ of points of holomorphy of $U(\lambda)$ in the full complex plane $\mathbb{C}$ and $\mathcal{E} \cap \mathcal{U}(J)$ will denote the class of entire J-inner mvf's.

If $U \in \mathcal{U}(I_m)$, then $U \in \mathcal{S}^{m\times m}$ and $U(\lambda)$ is said to be an $m\times m$ inner mvf. The set of $m\times m$ inner mvf's will be denoted $\mathcal{S}_{in}^{m\times m}$ and the set of outer $m\times m$

mvf's in $\mathcal{S}^{m\times m}$ will be denoted $\mathcal{S}^{m\times m}_{out}$. (It is perhaps useful to recall that if $s\in\mathcal{S}^{m\times m}$, then $s\in\mathcal{S}^{m\times m}_{out}$ if and only if det $s\in\mathcal{S}^{1\times 1}_{out}$, and that if $s\in\mathcal{S}^{1\times 1}$, then

$$s\in\mathcal{S}^{1\times 1}_{out}\Longleftrightarrow \ln|s(i)| = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln|s(\mu)|}{1+\mu^2}d\mu\,.)$$

If $A\in\mathcal{U}(J_p)$, there exists a pair of $p\times p$ inner mvf's $b_3(\lambda)$ and $b_4(\lambda)$ that are uniquely characterized in terms of the blocks of $B(\lambda)=A(\lambda)\mathfrak{V}$ and the set

$$\mathcal{N}^{p\times p}_{out} = \left\{\frac{g}{h} : g\in\mathcal{S}^{p\times p}_{out} \quad\text{and}\quad h\in\mathcal{S}^{1\times 1}_{out}\right\}$$

by the constraints

$$b_{21}^{\#}b_3\in\mathcal{N}^{p\times p}_{out} \quad\text{and}\quad b_4b_{22}\in\mathcal{N}^{p\times p}_{out}\,,$$

up to a constant $p\times p$ unitary multiplier on the left of $b_3(\lambda)$ and a constant $p\times p$ unitary multiplier on the right of $b_4(\lambda)$. Such a pair will be referred to as an *associated pair of the second kind* for $A(\lambda)$ and denoted

$$\{b_3, b_4\}\in ap_{II}(A)\,.$$

(There is also a set of associated pairs $\{b_1, b_2\}$ of the first kind that is more convenient to use in some other classes of problems that will not be discussed here.) The pairs $\{b_3^t, b_4^t\}\in ap_{II}(A_t)$ that are associated with the matrizant $A_t(\lambda)$, $0\le t<d$, of a canonical system of the form (1.1) with $J=J_p$ are entire $p\times p$ inner mvf's that are *monotonic* in the sense that

$$(b_3^{t_1})^{-1}b_3^{t_2} \quad\text{and}\quad b_4^{t_2}(b_4^{t_1})^{-1}$$

are $p\times p$ entire inner mvf's for $0\le t_1\le t_2<d$. Moreover, they are uniquely specified by imposing the *normalization* conditions $b_3^t(0)=b_4^t(0)=I_p$ for $0\le t<d$.

2. Reproducing kernel Hilbert spaces

If $U\in\mathcal{U}(J)$ and

$$\rho_\omega(\lambda) = -2\pi i(\lambda-\overline{\omega})\,,$$

then the kernel

$$K_\omega^U(\lambda = \frac{J-U(\lambda)JU(\omega)^*}{\rho_\omega(\lambda)}$$

is positive on $\mathfrak{h}_U\times\mathfrak{h}_U$ in the sense that $\sum_{i,j=1}^n u_i^*K_{\omega_j}^U(\omega_i)u_j\ge 0$ for every set of vectors $u_1,\dots,u_n\in\mathbb{C}^m$ and points $\omega_1,\dots,\omega_n\in\mathfrak{h}_U$; see [Dym 1989], for example. Therefore, by the matrix version of a theorem of Aronszajn [1950], there is an associated RKHS (reproducing kernel Hilbert space) $\mathcal{H}(U)$ of $m\times 1$ mvf's defined and holomorphic in $\mathfrak{h}_U$ with RK (reproducing kernel) $K_\omega^U(\lambda)$. This means that for every choice of $\omega\in\mathfrak{h}_U$, $u\in\mathbb{C}^m$ and $f\in\mathcal{H}(U)$,

(1) $K_\omega u \in \mathcal{H}(U)$ and
(2) $\langle f, K_\omega u\rangle_{\mathcal{H}(U)} = u^* f(\omega)$.

Thus, item (5) in the list of properties of the matrizant implies that there is a RKHS $\mathcal{H}(Y_t)$ of entire $m \times 1$ mvf's with RK $K_\omega^t(\lambda)$ for each $t \in [0, d)$, i.e., for each choice of $\omega \in \mathbb{C}$, $u \in \mathbb{C}^m$ and $f \in \mathcal{H}(Y_t)$,

(a) $K_\omega^t u \in \mathcal{H}(Y_t)$ and
(b) $\langle f, K_\omega^t u\rangle_{\mathcal{H}(Y_t)} = u^* f(\omega)$.

Moreover, property (6) of the matrizant Y_t implies that if $0 \le t_1 \le t_2 < d$, then $\mathcal{H}(Y_{t_1}) \subseteq \mathcal{H}(Y_{t_2})$ as sets and

$$\|f\|_{\mathcal{H}(Y_{t_2})} \leqslant \|f\|_{\mathcal{H}(Y_{t_1})}$$

for every $f \in \mathcal{H}(U_{t_1})$.

In this short review we shall restrict attention to canonical systems with signature matrices $J = J_p$ and shall denote the matrizant of such a system by $A_t(\lambda)$ and the corresponding RK by $K_\omega^{A_t}(\lambda)$. Thus,

$$K_\omega^{A_t}(\lambda) = \frac{J_p - A_t(\lambda) J_p A_t(\omega)^*}{\rho_\omega(\lambda)} .$$

Let

$$N_2^* = \sqrt{2}\,[\,0 \quad I_p\,] , \qquad B_t(\lambda) = A_t(\lambda)\mathfrak{V}$$

and

$$\mathfrak{E}_t(\lambda) = N_2^* B_t(\lambda) = [\,E_-(t, \lambda) \quad E_+(t, \lambda)\,]$$

with $p \times p$ components $E_\pm(t, \lambda)$. Then, since

$$N_2^* K_\omega^{A_t}(\lambda) N_2 = -\frac{\mathfrak{E}_t(\lambda) j_p \mathfrak{E}_t(\omega)^*}{\rho_\omega(\lambda)} ,$$

the kernel

$$K_\omega^{\mathfrak{E}_t}(\lambda) = -\frac{\mathfrak{E}_t(\lambda) j_p \mathfrak{E}_t(\omega)^*}{\rho_\omega(\lambda)} = \frac{E_+(t, \lambda) E_+(t, \omega)^* - E_-(t, \lambda) E_-(t, \omega)^*}{\rho_\omega(\lambda)}$$

is also positive and defines a RKHS of entire $p \times 1$ entire mvf's that we shall denote $\mathcal{B}(\mathfrak{E}_t)$. These spaces will be called *de Branges spaces*, since they were introduced and extensively studied by L. de Branges; see e.g., [de Branges 1968b; 1968a] and, for additional applications and expository material, [Dym and McKean 1976; Dym 1970; Dym and Iacob 1984]. They can be characterized in terms of the blocks $E_\pm(t, \lambda)$ by the following criteria:

$$f \in \mathcal{B}(\mathfrak{E}_t) \Longleftrightarrow (E_+^t)^{-1} f \in H_2^p \quad \text{and} \quad (E_-^t)^{-1} f \in K_2^p ,$$

where H_2^p denotes the vector Hardy space of order 2 and K_2^p denotes its orthogonal complement with respect to the standard inner product

$$\langle g, h\rangle_{st} = \int_{-\infty}^{\infty} h(\mu)^* g(\mu)\, d\mu \tag{2.1}$$

in $L_2^p(\mathbb{R})$. Moreover, if $f \in \mathcal{B}(\mathfrak{E}_t)$, then

$$\|f\|^2_{\mathcal{B}(\mathfrak{E}_t)} = \langle (E_+^t)^{-1} f, (E_+^t)^{-1} f\rangle_{st}\,.$$

It turns out that with each matrizant $A_t(\lambda)$, there is a unique associated pair $b_3^t(\lambda)$ and $b_4^t(\lambda)$ of $p \times p$ entire inner mvf's that meet the normalization conditions $b_3^t(0) = I_p$ and $b_4^t(0) = I_p$ and corresponding sets of RKHS's $\mathcal{H}(b_3^t)$ and $\mathcal{H}_*(b_4^t)$ with RK's

$$k_\omega^{b_3^t}(\lambda) = \frac{I_p - b_3^t(\lambda) b_3^t(\omega)^*}{\rho_\omega(\lambda)} \quad \text{and} \quad \ell_\omega^{b_4^t}(\lambda) = \frac{b_4^t(\lambda) b_4^t(\omega)^* - I_p}{\rho_\omega(\lambda)}\,,$$

respectively.

3. Linear fractional transformations

The linear fractional transformation T_U based on the four block decomposition

$$U(\lambda) = \begin{bmatrix} u_{11}(\lambda) & u_{12}(\lambda) \\ u_{12}(\lambda) & u_{22}(\lambda) \end{bmatrix},$$

of an $m \times m$ mvf $U(\lambda)$ that is meromorphic in $\mathbb{C}_+$ with diagonal blocks $u_{11}(\lambda)$ of size $p \times p$ and $u_{22}(\lambda)$ of size $q \times q$ is defined on the set

$$\mathcal{D}(T_U) = \{p \times q \text{ meromorphic mvf's } \varepsilon(\lambda) \text{ in } \mathbb{C}_+ \\ \text{such that } \det\{u_{21}(\lambda)\varepsilon(\lambda) + u_{22}(\lambda)\} \not\equiv 0 \text{ in } \mathbb{C}_+\}$$

by the formula

$$T_U[\varepsilon] = (u_{11}\varepsilon + u_{12})(u_{21}\varepsilon + u_{22})^{-1}.$$

If $U_1, U_2 \in \mathcal{U}(J)$ and if $\varepsilon \in \mathcal{D}(T_{U_2})$ and $T_{U_2}[\varepsilon] \in \mathcal{D}(T_{U_1})$ then

$$T_{U_1 U_2}[\varepsilon] = T_{U_1}[T_{U_2}[\varepsilon]]\,.$$

The notation

$$T_U[E] = \{T_U[\varepsilon] : \varepsilon \in E\} \quad \text{for } E \subseteq \mathcal{D}(T_U)$$

will be useful.

The principal facts are that

(1) $U \in \mathcal{U}(j_{pq}) \Longrightarrow \mathcal{S}^{p\times q} \subseteq \mathcal{D}_{T_U}$ and $T_U[\mathcal{S}^{p\times q}] \subseteq \mathcal{S}^{p\times q}$.
(2) $U \in \mathcal{U}(J_p) \Longrightarrow T_U[\mathcal{C}^{p\times p} \cap \mathcal{D}_{T_U}] \subseteq \mathcal{C}^{p\times p}$.

Moreover, if

$$B(\lambda) = A(\lambda)\mathfrak{V},$$

then

$$T_A[\mathscr{C}^{p\times p} \cap \mathfrak{D}(T_A)] \subseteq T_B[\mathscr{S}^{p\times p} \cap \mathfrak{D}(T_B)] \subseteq \mathscr{C}^{p\times p},$$

where the first inclusion may be proper. The set

$$\mathscr{C}(A) = T_B[\mathscr{S}^{p\times p} \cap \mathfrak{D}(T_B)].$$

is more useful than the set $T_A[\mathscr{C}^{p\times p} \cap \mathfrak{D}(T_A)]$.

We remark that

$$\mathscr{S}^{p\times p} \subseteq \mathfrak{D}(T_B) \iff b_{22}(\omega)b_{22}(\omega)^* > b_{21}(\omega)b_{21}(\omega)^* \tag{3.1}$$

for some (and hence every) point $\omega \in \mathfrak{h}_A^+$; see Theorem 2.7 in [Arov and Dym 2003a].

4. Restrictions and consequences

In addition to fixing the signature matrix $J = J_p$ in the canonical system (1.1), we shall assume that $\mathscr{H}(A_t) \subset L_2^m$ for every $t \in [0, d)$, i.e., (in our current terminology) A_t belongs to the class $\mathscr{U}_{rsR}(J_p)$ of *right strongly regular J-inner mvf's*. (In our earlier papers the set $\mathscr{U}_{rsR}(J_p)$ was designated $\mathscr{U}_{sR}(J_p)$.) One of the important consequences of this assumption rests on the fact that

$$\mathscr{H}(A_t) \subset L_2^m \iff \mathscr{C}(A_t) \cap \mathring{\mathscr{C}}^{p\times p} \neq \varnothing, \tag{4.1}$$

where

$$\mathring{\mathscr{C}}^{p\times p} = \{c \in \mathscr{C}^{p\times p} : c \in H_\infty^{p\times p} \quad \text{and} \quad \Re c(\mu) \geq \delta_c I_p \quad \text{a.e. on } \mathbb{R}\} \tag{4.2}$$

and $\delta_c > 0$. Other characterizations of the class $\mathscr{U}_{rsR}(J)$ in terms of the Treil–Volberg matrix version of the Muckenhoupt $(A)_2$ condition presented in [Treil and Volberg 1997] are furnished in [Arov and Dym 2001] and [Arov and Dym 2003b].

If $A_t \in \mathscr{U}_{rsR}(J_p)$ for every $t \in [0, d)$, the following conclusions are in force:

(1) The unique normalized monotonic chain of $p \times p$ entire inner mvf's

$$\{b_3^t, b_4^t\} \in ap_{II}(A_t)$$

consists of continuous functions of t on the interval $0 \leq t < d$ for each fixed point $\lambda \in \mathbb{C}$.

(2) The RKHS $\mathscr{H}(A_{t_1})$ is isometrically included in the RKHS $\mathscr{H}(A_{t_2})$ for $0 \leq t_1 \leq t_2 < d$.

(3) The de Branges spaces $\mathcal{B}(\mathfrak{E}_t)$ based on the de Branges matrix

$$\mathfrak{E}_t(\lambda) = \sqrt{2}\,[0 \quad I_p\,]\,A_t(\lambda)\mathfrak{V}\,, \quad \text{for } 0 \le t < d$$

are nested by isometric inclusion, i.e., $\mathcal{B}(\mathfrak{E}_{t_1})$ is isometrically included in $\mathcal{B}(\mathfrak{E}_{t_2})$ for $0 \le t_1 \le t_2 < d$.

(4) $\mathcal{B}(\mathfrak{E}_t) = \mathcal{H}(b_3^t) \oplus \mathcal{H}_*(b_4^t)$ as Hilbert spaces with equivalent norms.

(5) The mapping $N_2^* : f \in \mathcal{H}(A_t) \longrightarrow N_2^* f \in \mathcal{B}(\mathfrak{E}_t)$ is unitary for every $t \in [0, d)$.

(6) If $\mathcal{S}^{p\times p} \subseteq \mathcal{D}(T_{B_{t_0}})$ for some $t_0 \in [0, d)$, then:

(a) $\mathcal{S}^{p\times p} \subseteq \mathcal{D}(T_{B_t})$ for every $t \in [t_0, d)$.

(b) $\mathcal{C}(A_{t_2}) \subseteq \mathcal{C}(A_{t_1})$ for $t_0 \le t_1 \le t_2 < d$.

(c) $\bigcap_{t_0 \le t < d} \mathcal{C}(A_t) \ne \varnothing$.

(d) $\{c(\omega) : \omega \in \mathbb{C}_+ \text{ and } c \in \bigcap_{t_0 \le t < d} \mathcal{C}(A_t)\}$ is a (Weyl–Titchmarsh) matrix ball with left and right semiradii $R_\ell(\omega)$ and $R_r(\omega)$ with

$$\operatorname{rank} R_\ell(\omega) = \operatorname{rank}\{\lim_{t\uparrow\infty} b_3^t(\omega) b_3^t(\omega)^*\} \tag{4.3}$$

and

$$\operatorname{rank} R_r(\omega) = \operatorname{rank}\{\lim_{t\uparrow\infty} b_4^t(\omega)^* b_4^t(\omega)\}\,. \tag{4.4}$$

Moreover, these ranks are independent of the choice of the point $\omega \in \mathbb{C}_+$.

An mvf $c(\lambda)$ that belongs to the set

$$\mathcal{C}_{\text{imp}}(H) = \bigcap_{t_0 \le t < d} \mathcal{C}(A_t)$$

is called an input impedance (or Weyl function) of the system (1.1). If $H \in L_1^{m\times m}$, then, without loss of generality, it may be assumed that $d < \infty$. In this case, $A_d(\lambda)$ is the monodromy matrix of the system (1.1), $\mathcal{C}_{\text{imp}}(H) = \mathcal{C}(A_d)$ and the semiradii $R_\ell(\omega)$ and $R_r(\omega)$ are both positive definite.

5. Inverse problems for canonical systems

Inverse problems for the canonical system (1.1) aim to recover $H(t)$, given some information about the solution of the system. In this direction it is convenient to first consider inverse problems for the canonical integral system

$$y(t, \lambda) = y(0, \lambda) + i\lambda \int_0^t y(s, \lambda)\, dM(s)\, J_p \quad \text{for } 0 \le t < d\,, \tag{5.1}$$

in which the mass function $M(t)$, $0 \le t < d$ is a continuous nondecreasing $m \times m$ mvf on the interval $[0, d)$ with $M(0) = 0$. Then the matrizant $A_t(\lambda) = A(t, \lambda)$ of this system is a continuous solution of the equation

$$A_t(\lambda) = I_m + i\lambda \int_0^t A_s(\lambda)\, dM(s)\, J_p \quad \text{for } 0 \le t < d \tag{5.2}$$

and consequently,

$$\frac{\partial A_t}{\partial \lambda}(\lambda)\Big|_{\lambda=0} = \lim_{\lambda \to 0} \frac{A_t(\lambda) - I_m}{\lambda} = iM(t)J_p\,. \tag{5.3}$$

Thus, $M(t)$, $0 \le t < d$, can be recovered from the matrizant $A_t(\lambda)$, $0 \le t < d$. The main tool is the following result, which, for a given triple $b_3 \in \mathcal{S}_{in}^{p\times p}$, $b_4 \in \mathcal{S}_{in}^{p\times p}$ and $c \in \mathcal{C}^{p\times p}$, is formulated in terms of the sets

$$\mathcal{C}(b_3, b_4; c) = \{\widetilde{c} \in \mathcal{C}^{p\times p} : b_3^{-1}(\widetilde{c} - c)b_4^{-1} \in \mathcal{N}_+^{p\times p}\}$$

and

$$\mathcal{N}_+^{p\times p} = \left\{\frac{g}{h} : g \in \mathcal{S}^{p\times p} \quad \text{and} \quad h \in \mathcal{S}_{out}^{1\times 1}\right\}.$$

The set $\mathcal{C}(b_3, b_4; c)$ was identified as the set of solutions of a generalized Carathéodory interpolation problem that is formulated in terms of the three given mvf's b_3, b_4 and c in [Arov 1993] and connections with the class $\mathcal{U}(J_p)$ were studied there. These results were developed further in [Arov and Dym 1998] in the case that $b_3(\lambda)$ and $b_4(\lambda)$ are also entire mvf's. In that special case, the interpolation problem is equivalent to a bitangential Krein extension problem in a class of helical mvf's. Krein understood the deep connections between such extension problems and inverse problems for canonical systems. Theorem 5.2, below, illustrates the Krein strategy of identifying the solution of an inverse problem with an appropriately defined chain of extension problems; see [Arov and Dym 2005b] for more details.

THEOREM 5.1. *Let $b_3(\lambda)$, $b_4(\lambda)$ be a pair of entire $p \times p$ inner mvf's and let $c \in \mathcal{C}^{p\times p}$. Then there exists at most one mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ such that*

(1) $\mathcal{C}(A) = \mathcal{C}(b_3, b_4; c)$.
(2) $\{b_3, b_4\} \in ap_{II}(A)$.
(3) $A(0) = I_m$.

Moreover, if

$$\mathcal{C}(b_3, b_4; c) \cap \mathring{\mathcal{C}}^{p\times p} \neq \varnothing\,,$$

there exists exactly one mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ for which these three conditions are met and it is automatically right strongly regular.

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Correspondingly, in our formulation of the inverse impedance problem (inverse spectral problem) for the canonical integral system (5.1) we shall specify a continuous monotonic normalized chain of entire inner $p \times p$ entire inner mvf's $\{b_3^t(\lambda), b_4^t(\lambda)\}$, in addition to an mvf $c \in \mathcal{C}^{p\times p}$ (or a spectral function $\sigma(\mu)$). Spectral functions and the inverse spectral problem are introduced in Section 9.

THEOREM 5.2. *Let $\{b_3^t(\lambda), b_4^t(\lambda)\}, 0 \le t < d$, be a normalized monotonic continuous chain of pairs of entire inner $p \times p$ mvf's and let $c \in \mathcal{C}^{p\times p}$. Then there exists at most one Hamiltonian $H(t)$, $0 \le t < d$, such that the matrizant $A_t(\lambda)$ of the corresponding canonical system meets the following conditions for every $t \in [0, d)$:*

(1) $\mathcal{C}(A_t) = \mathcal{C}(b_3^t, b_4^t; c)$.
(2) $\{b_3^t, b_4^t\} \in ap_{II}(A_t)$.
(3) $A_t(0) = I_m$.

There exists exactly one continuous nondecreasing mvf $M(t)$ on the interval $[0, d)$ with $M(0) = 0$ such that the matrizant A_t of the integral system (5.1) *meets these conditions if*

$$\mathcal{C}(b_3^t, b_4^t; c) \cap \mathring{\mathcal{C}}^{p\times p} \neq \varnothing \text{ for every } t \in [0, d)\ . \tag{5.4}$$

PROOF. See Theorem 7.9 in [Arov and Dym 2003a]. □

6. Description of the RKHS's $\mathcal{H}(A)$ and $\mathcal{B}(\mathfrak{E})$ for $A \in \mathcal{U}_{rsR}(J_p)$

THEOREM 6.1. *If $A \in \mathcal{U}_{rsR}(J_p)$, $\{b_3(\lambda), b_4(\lambda)\} \in ap_{II}(A)$ and $c \in \mathcal{C}(A) \cap H_\infty^{p\times p}$, then*

$$\mathcal{H}(A) = \left\{ \begin{bmatrix} -\Pi_+ c^* g + \Pi_- c h \\ g + h \end{bmatrix} : g \in \mathcal{H}(b_3) \ \text{ and } \ h \in \mathcal{H}_*(b_4) \right\},$$

where Π_+ denotes the orthogonal projection of L_2^p onto the Hardy space H_2^p, $\Pi_- = I - \Pi_+$ denotes the orthogonal projection of L_2^p onto $K_2^p = L_2^p \ominus H_2^p$,

$$\mathcal{H}(b_3) = H_2^p \ominus b_3 H_2^p \quad \text{and} \quad \mathcal{H}_*(b_4) = K_2^p \ominus b_4^* K_2^p\ .$$

Moreover,

$$f = \begin{bmatrix} -\Pi_+ c^* g + \Pi_- c h \\ g + h \end{bmatrix} \Longrightarrow \langle f, f \rangle_{\mathcal{H}(A)} = \langle (c + c^*)(g + h), g + h \rangle_{st}\ ,$$

where $g \in \mathcal{H}(b_3)$, $h \in \mathcal{H}_*(b_4)$ and $\langle \cdot, \cdot \rangle_{st}$ denotes the standard inner product (2.1) *in L_2^p.*

PROOF. See Theorem 3.8 in [Arov and Dym 2005a]. □