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Geometry in Curvature Theory

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ABSTRACT. This article is based on the Roever Lectures in Geometry given by Kuiper at Washington University, St. Louis, in January 1986. Although incomplete, it is an excellent exposition of the topics it does cover, starting with elementary versions of the notion of tightness and going through the analysis of topsets, the classification in low dimensions, the notions of total curvature for curves and surfaces in space, homological notions of tightness, the Morse inequalities, and Poincaré polynomials. It contains a detailed proof of Kuiper's remarkable result that a tight two-dimensional surface substantially immersed in $\mathbb{R}^5$ must be a Veronese surface.

EDITORS' NOTE. At the time of Kuiper's death in December, 1994, this paper existed in the form of an unfinished typescript. For inclusion in this volume, it was edited by Thomas Banchoff, Thomas Cecil, Wolfgang Kühnel, and Silvio Levy. A few Editors' Notes such as this one were included, mostly pointing to additional references. Several minor typos were corrected and the numbering was normalized for ease of reference; thus Sections 5 and 6 of the manuscript were renumbered 4 and 5, since there was no Section 4. The present illustrations were made by Christine Heinitz and by Levy, based on Kuiper's hand drawings.

1. Banchoff's two-piece property. Zero-tightness

Prerequisites and Notation. *Euclidean space* $E = E^N$ of dimension N is the real vector space $\mathbb{R}^N$ with norm $\|u\| = \sqrt{\Sigma_1^N (u^i)^2}$ for $u = (u^1, \dots, u^N) \in \mathbb{R}^N$ and distance $\|v - u\|$ for $u, v \in \mathbb{R}^N$. The identification $\kappa : \mathbb{R}^N \to E^N$ can be replaced by any other preferred Euclidean coordinate system $\kappa \circ g : \mathbb{R}^N \to E^N$, where g is an isometry:

$$g(u) = u_0 + u \cdot g_0, \quad \text{for } u_0 \in \mathbb{R}^N \text{ and } g_0 \in O(N) \text{ an orthogonal matrix.}$$

We use E^N to emphasize Euclidean space aspects and $\mathbb{R}^N$ for vector space aspects. A set $X \subset E$ is called *convex* if

$$u + \lambda(v - u) \in X \quad \text{for all } u, v \in X \text{ and } 0 \le \lambda \le 1.$$

The smallest convex set containing $X \subset E$ is its *convex hull*, denoted $\mathcal{H}X$. The smallest affine subspace (also a Euclidean space) that contains X is its *span*, denoted span(X). If span$(X) = E$, then X is called *substantial* in E. The boundary of $\mathcal{H}X$ in span(X) is called the *convex envelope* $\partial\mathcal{H}X$ of $X \subset E$. If X is one point then $\mathcal{H}X = X$ and $\partial\mathcal{H}X = \varnothing$.

The subspaces $\{u : \|u\| < r\}$ and $\{u : \|u\| = r\}$, for $r > 0$, are called the *N-ball* B^N and the *$(N-1)$-sphere* S^{N-1}, respectively. As metric spaces they are called the *round ball* and the *round sphere*. Let $z : \mathbb{R}^N \to \mathbb{R}$ be a *linear function*, $z(u) = \sum_i \zeta_i u^i$ for $\zeta_i \in \mathbb{R}$, with $\|z\|^2 = \sum \zeta_i^2 > 0$. The subspaces $\{u : z(u) \geq c\}$, $\{u : z(u) > c\}$ and $\{u : z(u) = c\}$ of E are called the *half-space* h, its interior the *open half-space* $\mathring{h}$, and its boundary the *hyperplane* ∂h, respectively. The function z is often called a *height function.*

A metrizable topological space X is called *separated* if it is the disjoint union of two nonempty open and closed subsets, say X_1 and X_2. If $U \subset X$ contains points $x_1 \in X_1$ and $x_2 \in X_2$, then $U \cap X_1$ and $U \cap X_2$ are disjoint open and closed in U, and so U is also separated. The space X is called *connected* if it is not separated. A connected nonempty open closed subset of a metrizable space X is called a *topological component* of X.

EXAMPLE. The plane set

$$\{(\xi, \eta) : \xi = 0 \text{ or } \eta = \sin(\xi^{-1})\}$$

is connected (but not pathwise connected).

CONSEQUENCE. If Y is a metrizable space and for any two points $y_1, y_2 \in Y$ there is a connected space $W(y_1, y_2) \subset Y$ containing y_1 and y_2 (in other words, if "any two points $y_1, y_2 \in Y$ can be connected in Y"), then Y is connected. Indeed Y separated would show an immediate contradiction.

Definitions and General Theorems. For given compact spaces X, either embedded in $E = E^N$ or given independently, we are interested in embeddings or other continuous maps in E with nice properties that generalize convexity. We introduce the important notion and tool called a topset:

DEFINITION. Suppose the half-space

$$h = h_z = \{u \in E : z(u) \geq c\}$$

for some linear function $z : E \to \mathbb{R}$ supports ("leans against") the compact set X, without containing it completely:

$$X \neq h \cap X = \partial h \cap X \neq \varnothing.$$

Then $X_z = h \cap X$ is called a (proper) *topset* of the set $X \subset E$. It is the set of points in X for which the function $z : X \to \mathbb{R}$ attains its maximal (top) value.

Note that $\mathring{h} \cap X = \varnothing$. More generally, if $f : X \to E$ is a continuous map of the compact space X into E, and $h \cap f(X)$ is a topset of $f(X) \subset E$, then

$$f^{-1}(h) = \{x \in X : f(x) \in h\} = f^{-1}(\partial h) \subset X$$

is called a *topset of the map* f. A topset of a topset is called a *top*^{2}*set*. A top^{j}set for some $j \geq 1$ is called a *top***set*. If the span of a top*set X' of X has dimension k, then X' is called an E^k-*top***set*.

REMARK. Let X be a substantial set of $N+1$ points $e_0, \dots, e_N$ in E^N. Then $\mathcal{H}X$, the convex hull, is an N-simplex σ_N. Any proper nonempty subset of X is a topset.

EXERCISE. Determine all topsets of a standard torus in E^3, obtained by rotating a circle around a disjoint line in its plane.

EXERCISE. Determine the topsets of the map $f : w \to w^3$ of the unit circle $\{w : |w| = 1\} \subset \mathbb{C}$ into $\mathbb{C} = \mathbb{R} \oplus \mathbb{R} = E^2$.

THEOREM 1.1. *The convex envelope of a compact set $X \subset E$ is the union of the convex hulls of its topsets*:

$$\partial \mathcal{H} X = \bigcup_z \mathcal{H} X_z.$$

The union may be taken only over all linear functions $z : E \to \mathbb{R}$ with norm $\|z\| = 1$. If X consists of one point, both sides are the empty set.

PROOF. We need to show the implications in both directions:

$$x \in \bigcup_z \mathcal{H} X_z \iff x \in \partial \mathcal{H} X.$$

Assume that the span of X is $\operatorname{span}(X) = E = E^N$. For a topset $X_z = h \cap X$, let $x \in \mathcal{H} X_z$. Since h supports X, it supports also $\mathcal{H}X$. Then $x \in h \cap \mathcal{H}X \subset \partial \mathcal{H}X$. Conversely if $x \in \partial \mathcal{H}X$, then there is a $\mathcal{H}X$-supporting half-space h_z containing x, and

$$x \in h_z \cap \partial \mathcal{H}X = h_z \cap \mathcal{H}X = h_z \cap \mathcal{H}X_z = \mathcal{H}X_z. \qquad \square$$

Now we propose a preliminary generalization of convexity:

DEFINITION (for compact sets). A connected compact set $X \subset E$ is said to have the *two-piece property*, or TPP [Banchoff 1971b], and is called 0-*tight*, in case any of the following equivalent conditions hold:

(a) $h \cap X$ is connected for every half-space h.
(b) $\mathring{h} \cap X$ is connected for every h.
(c) The set difference $X \setminus \partial h$ has at most two components for every h (this is the two-piece property).
(d) In terms of Čech homology and any coefficient ring, the homomorphism $H_0(h \cap X) \to H_0(X)$ is injective for every h.

We mention this last condition now for the sake of completeness, but defer the relevant discussion till later (page 35).

Proof of equivalence. (a) $\Rightarrow$ (b). If (a) holds then any two points in $\mathring{h} \cap X$ are contained for some half-space h_i in $h_i \cap X \subset \mathring{h} \cap X$, and they can be connected in $h_i \cap X$. Then $\mathring{h} \cap X$ is connected.

(b) $\Rightarrow$ (a). Suppose $h \cap X$ is not connected for some $h = \{u \in E : z(u) \geq c\}$, let and Y_1 and Y_2 be disjoint nonempty open closed subsets with union $Y_1 \cup Y_2 = h \cap X$. Let U_1 and $U_2 \subset X$ be disjoint nonempty open neighborhoods of the compact subsets Y_1 and Y_2. If $c - 2\varepsilon$ is the maximum of z on the compact set $X \setminus (U_1 \cup U_2)$ and $\mathring{h}_0 = \{u \in E : z(u) > c - \varepsilon\}$, then $\mathring{h}_0 \cap X$ is not connected. This contradicts (b).

The equivalence (b) – (c) is tautological. □

The same proof works for the equivalences in the following more general situation.

Definition (for maps). A continuous map $f : X \to E$ of a connected compact space X in E has the *two-piece-property* (TPP) and is called 0-*tight* if any of the following equivalent conditions hold:

(a) $f^{-1}(h)$ is connected for any half-space h.
(b) $f^{-1}(\mathring{h})$ is connected for any h.
(c) $f^{-1}(E \setminus \partial h)$ has at most two components for any h (two-piece-property).
(d) $H_0(f^{-1}(h)) \to H_0(X)$ is injective for any h.

Examples. The following are 0-tight sets:

(1) a convex body $X = \mathcal{H}X \subset E^N$, for $N \geq 0$;
(2) a convex hypersurface $X = \partial\mathcal{H}X$ substantial in E^N, for $N \geq 2$ (convex curve for $N = 2$);
(3) a hemisphere, $\{u \in \mathbb{R}^3 : \|u\| = 1,\ z(u) \leq 0\}$;
(4) the standard round torus in E^3 (see the first exercise on page 3);
(5) the solid round ring (solid torus) bounded by the standard torus;
(6) the 1-*skeleton* $\mathrm{Sk}_1(\sigma_N)$ of the N-simplex $\sigma_N \subset E^N$; this is by definition the union of all edges of σ_N, and as a topological space it is a *complete graph* on $N + 1$ vertices.

Remark. These are corollaries of the definition:

(1) 0-tightness is invariant under *linear embeddings* $i : \mathbb{R}^M \to \mathbb{R}^N$ and *projections* $p : \mathbb{R}^N \to \mathbb{R}^M$, where $M < N$. Indeed, if $f : X \to \mathbb{R}^M$ and $g : Y \to \mathbb{R}^N$ are 0-tight, then so are $i \circ f : X \to E^N$ and $p \circ g = Y \to E^M$.

An example of a 0-tight map (immersion) is the projection of the 1-skeleton $\mathrm{Sk}_1(\sigma_3)$ in E^3 onto the union of edges and diagonals of a convex 4-gon in a plane in E^3. Note that $f : X \to$ point $\in E^N$ is 0-tight for any connected compact X.

(2) 0-tightness is an *affine* and even a *projective property* in the following sense. Let P^N be a real projective N-space and P^{N-1} a hyperplane. Then $P^N \setminus P^{N-1}$

can be identified with E^N, and this identification is natural up to affine transformations $u \mapsto u_0 + gu$, where $u_0 \in \mathbb{R}^N$, $g \in \mathrm{GL}(n, \mathbb{R})$. Given $X \subset E^N$, let $\eta : P^N \to P^N$ be a projective transformation such that $\eta(X) \subset E^N \subset P^N$. Suppose $f : X \to E^N$ is 0-tight. Then also $\eta \circ f : X \to E^N$ is 0-tight by condition (c), which is expressed in terms of hyperplane sections.

THEOREM 1.2. *Any topset X_z of a 0-tight set $X \subset E$ or of a 0-tight map $f : X \to E$ is itself 0-tight. So is any top*set.*

PROOF. We deal with the case of a set; the proof for a map is the same. Suppose $X \subset E$ has a topset that is not 0-tight, say $X_z = h_z \cap X$, where $h_z = \{u \in E : z(u) \geq c_1\}$. See Figure 1. Then there exists a half-space $h_0 = \{u \in E : w(u) \geq c_2\}$ such that

$$\varnothing \neq h_0 \cap X_z \neq X_z,$$

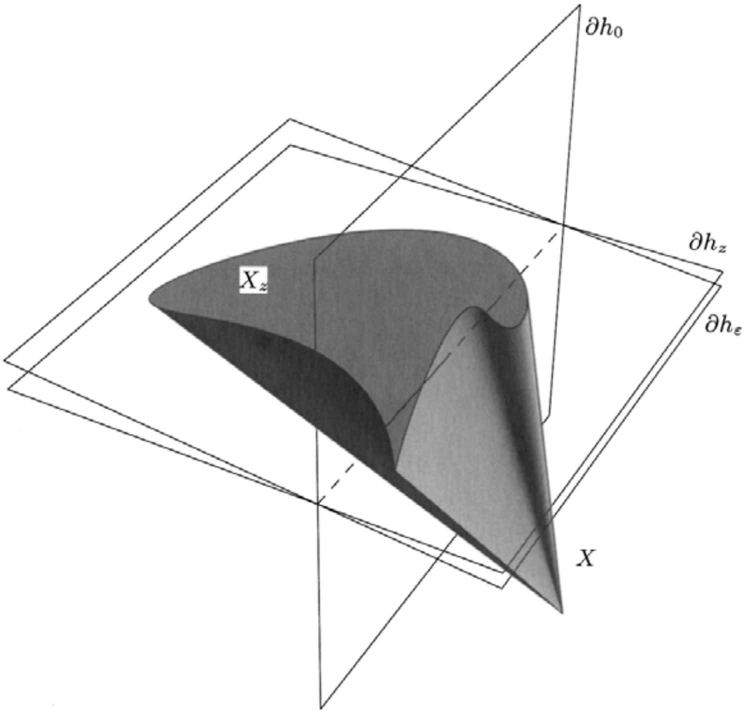

Figure 1. If a topset X_z of X is not 0-tight, some hyperplane ∂h_0 cuts X_z into more than two pieces. By tilting slightly the support hyperplane ∂h_z of X_z around the hinge $\partial h_z \cap \partial h_0$, one can obtain a hyperplane that cuts X into more than two pieces, so X is not 0-tight. (In the figure, the half-space h_0 is to the right of, and the half-spaces h_z and h_ε are above, their bounding planes.)

and $h_0 \cap X_z = h_0 \cap h_z \cap X = X_1 \cup X_2$ is separated with X_1 and X_2 open and closed in the relative topology, compact, and disjoint. There is a (small) open neighborhood U in X of $X_1 \cup X_2$ that is also separated:

$$U = U_1 \cup U_2, \quad U_1 \cap U_2 = \varnothing, \quad U_1 \supset X_1, \quad U_2 \supset X_2.$$

For small ε, the half-space

$$h_\varepsilon = \{u \in E : (z(u) - c_1) + \varepsilon(w(u) - c_2) \geq 0\}$$

meets X in $h_\varepsilon \cap X \supset h_0 \cap h_z \cap X$ and

$$h_\varepsilon \cap X \subset U = U_1 \cup U_2.$$

Therefore $h_\varepsilon \cap X$ is not connected, so X is not 0-tight. □

THEOREM 1.3 (CLASSIFICATION OF PLANE 0-TIGHT SETS). *Any plane compact 0-tight substantial set $X \subset E^2$ can be obtained from its convex hull $\mathcal{H}X$ by deleting a countable family of disjoint open convex subsets from the interior:*

$$X = \mathcal{H}X \setminus \bigcup_{i=0}^{r} U_i, \quad \textit{where } 0 \leq r \leq \infty.$$

PROOF. Let X be substantial and 0-tight in E^2. Every topset X_z of X is 0-tight and lies in a supporting line ∂h_z, so it is either a point or a line segment, and in any case convex. By Theorem 1.1 we have $\bigcup_z X_z = \bigcup_z \mathcal{H}X_z = \partial\mathcal{H}X$, and $\partial\mathcal{H}X$ is contained in X. The set X is then obtained from $\mathcal{H}X$ by deleting disjoint open connected sets (holes) U. Any embedded circle in a hole U does not separate X, and can be contracted inside U to a point. So each hole U is contractible and homeomorphic to an open disc.

Now suppose U is not convex, so there are distinct $u_1, u_2 \in U$ and $0 < \lambda < 1$ such that $u_1 + \lambda(u_2 - u_1) \notin U$. The smallest such value λ for given u_1 and u_2 yields a point $x = u_1 + \lambda(u_2 - u_1)$ in X. Let h and h^- be the two half-planes having as common boundary the line $u_1u_2 = \partial h = \partial h^-$. Connect u_1 and u_2 by an embedded polygonal arc $\beta \subset U$ that meets the line ∂h transversally in every intersection point (see Figure 2). Since X is 0-tight, $h \cap X$ and $h^- \cap X$ are connected. Then x and $h \cap \partial\mathcal{H}X$ can be connected in $h \cap X$ and (even better) in the component of $(h \cap \mathcal{H}X) \setminus \beta$, which contains $h \cap X$. In $h \cap X$, the points x and $h \cap \partial\mathcal{H}X$ can be connected by a polygonal arc α, which meets ∂h only in the point x.

There is another such polygonal arc α^- in $(h^- \cap \mathcal{H}X) \setminus \beta$ connecting x with $h^- \cap \partial\mathcal{H}X$. The union $\alpha \cup \alpha^-$ lies in $\mathcal{H}X \setminus \beta$ and divides the segment $\partial h \cap \mathcal{H}X$, as well as $\mathcal{H}X$, into two parts, one containing u_1 and the other containing u_2. This contradicts the existence of the arc β from u_1 to u_2. So all holes U are open convex discs.

Any collection of disjoint open sets in the plane is countable. □

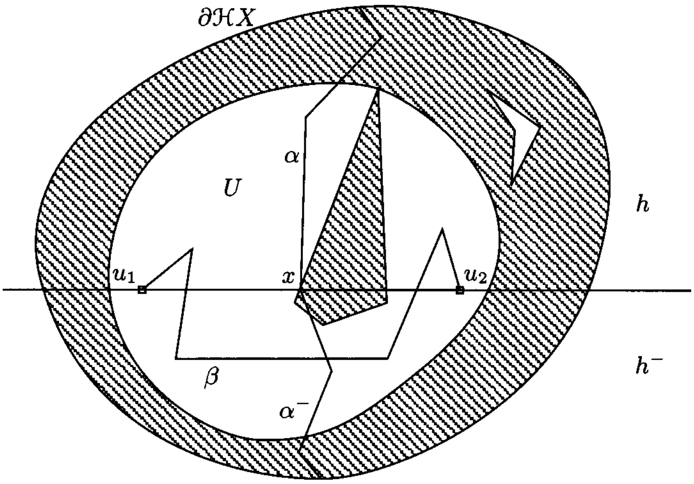

Figure 2. The holes in a 0-tight plane set must be convex (see the proof of Theorem 1.3).

Example. For later reference we mention a curious example of 0-tight plane set, the *limit Swiss cheese.* A Swiss cheese is a round 2-ball (disc) in E^2 from which a union of disjoint open round discs is deleted; see Figure 3. Touching of discs in their boundaries is permitted. If the union of the open discs is everywhere dense, the resulting 0-tight set is called a *limit Swiss cheese.*

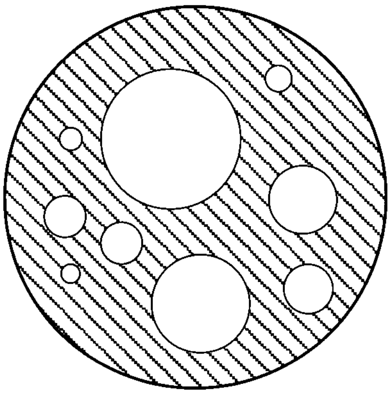

Figure 3. A Swiss cheese has the TPP.

Definition. The subspace $Y \subset X \subset E^N$ is a *local topset* of X if Y has an open neighborhood U in X such that Y is a topset of $\bar{U} \subset E$ (here as elsewhere the bar indicates closure). That means

$$X \neq Y = h \cap \bar{U} = \partial h \cap \bar{U} \neq \varnothing$$

for some half-space h. Note that Y is then open and closed in $\partial h \cap X$.

THEOREM 1.4 (TOPSETS). *The connected compact set $X \subset E$ is 0-tight if and only if every local topset of X is a connected topset of X; equivalently, if and only if every height function z has one (connected) maximum on X.*

The generalization for maps is as follows:

DEFINITION. The subset $Y \subset X$ is a *local topset* of the map $f : X \to E$ if Y has an open neighborhood U in X such that Y is a topset of the restriction $f|_{\bar{U}} : \bar{U} \to E$, that is

$$X \neq Y = (f|_{\bar{U}})^{-1}h = (f|_{\bar{U}})^{-1}(\partial h) \neq \varnothing,$$

for some half-space h.

THEOREM 1.4 (MAPS). *Let X be connected and compact. The map $f : X \to E$ is 0-tight if and only if every local topset Y of f is a connected topset of f; that is, if and only if every z has one connected maximum on X.*

REMARKS. If the continuous function $f : X \to \mathbb{R}$ has exactly one local topset, and so does the function $-f$, then f is 0-tight. The map $z \mapsto z^3$ for $|z| = 1$ (see the second exercise on page 3) is not 0-tight.

PROOF OF THEOREM 1.4 FOR SETS. If X is not 0-tight, there is a half-space $h' = \{u \in E : z(u) \geq c'\}$ for which $h' \cap X$ is separated and is the disjoint union of two open closed subsets X_1 and X_2. Let $c \geq c'$ be the maximal value for which $h \cap X_1$ and $h \cap X_2$ are both nonempty, where $h = \{u \in E : z(u) \geq c\}$. One at least of $h \cap X_1$ and $h \cap X_2$ is then a local topset and not a connected topset.

Conversely, if $Y \subset X$ is a local topset in $\partial h \cap X$ and $\partial h \cap X$ is not a connected topset, then $\partial h \cap X = Y \cup Z$ is the disjoint union of Y and Z and $h \cap X$ is the disjoint union of Y and $h \cap X \setminus Y \supset Z$, both open and closed. So $h \cap X$ is separated and X is not 0-tight. □

EXERCISE. Prove Theorem 1.4 for maps.

EXAMPLE. Let $\sigma_4 = \mathcal{H}(\{e_1, \dots, e_5\}) \subset E^4$ be a four-simplex, and let M be the union of five triangles $\mathcal{H}(\{e_i, e_{i+1}, e_{i+2}\})$, for $i = 1, \dots, 5$ (indices being taken modulo 5). Then M is a 0-tight Möbius band, substantial in E^4. Observe that M contains the 1-skeleton $\mathrm{Sk}_1(\sigma_4)$. Figure 4 shows a projection in E^3 (a 0-tight embedding), as well as a projection in E^2 (a 0-tight map) with folds along the edges e_1e_2, e_2e_3, e_3e_4, e_4e_5, and e_5e_1. The boundary of M is the polygon $e_1e_3e_5e_4e_2e_1$. To prove that $M \subset E^4$ is 0-tight, we observe that any local topset Y of M contains at least one vertex and cannot cut any opposite edge in σ_N transversally. Then Y lies in a supporting half-space h and $Y = h \cap M = \partial h \cap M$.

REMARK. For the same reason any union X of $\mathrm{Sk}_1(\sigma_N) \subset E^N$ with some of the simplices of σ_N of various dimensions is 0-tight. In particular, $\mathrm{Sk}_i(\sigma_N)$ is 0-tight, for $1 \leq i \leq N$.

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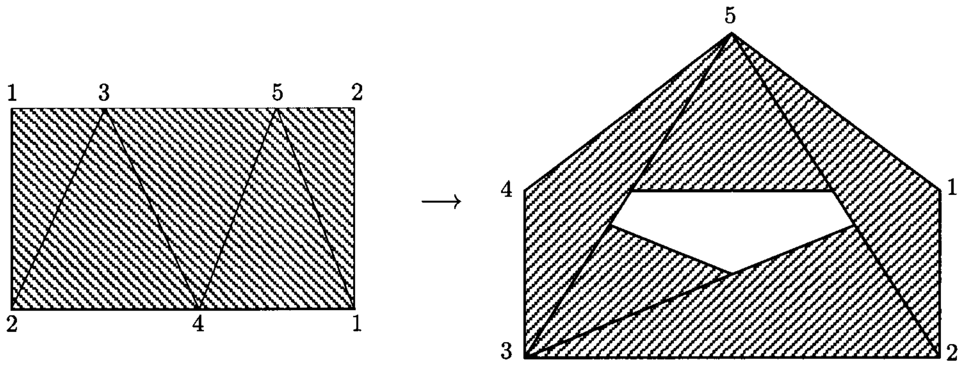

Figure 4. A Möbius band made from five of 2-faces of a tetrahedron. The embedded in E^3 is 0-tight, as is the projection onto the plane of the paper.

Zero-Tight Balls and Spheres of Dimension 1 and 2. We now prove some classification theorems with the help of our tool, the topsets.

A 0-tight embedded arc (1-ball) $X \subset E^N$ in E^N is necessarily a straight line segment. If not, there is a point $y \in X$ not on the line connecting the endpoints x_0 and x_1, and a half-space h containing x_0 and x_1 but not y. So $h \cap X$ is separated and X is not 0-tight.

Theorem 1.5 (0-tight circles, spheres, balls).

(i) *A 0-tight embedded closed curve in E^N is a plane convex curve.*
(ii) *A 0-tight substantial 2-sphere in E^N is a convex surface $X = \partial\mathcal{H}X$ in 3-space E^3.*
(iii) [Lastufka 1981] *A 0-tight substantial 2-ball (disc) in E^N is either*
  (a) *a convex plane disc in E^2, or*
  (b) *$X = \partial\mathcal{H}X \setminus U$ in E^3, where the deleted set U is an open disc of a plane convex topset $(\partial\mathcal{H}X)_z = \mathcal{H}X_z$* (see Figure 5).

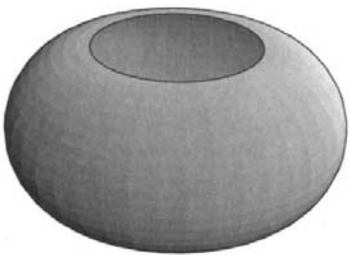

Figure 5. Zero-tight disc in E^3.

Proof. If $X \subset E$ is 0-tight, then by Theorem 1.2 so are any of its top*sets. An E_0-top*set of X is a point; an E_1-top*set of X is a line segment; possible E_2-top*sets of X are described in Theorem 1.3.

(i) If X is a 0-tight closed curve, or "circle" for short, in E^2, then

$$\partial\mathcal{H}X = \bigcup_z \mathcal{H}X_z = \bigcup_z X_z$$

is a circle embedded in X, hence equal to X, and X is a plane convex curve.

Now let X be a 0-tight substantial circle X in E^N, where $N \geq 3$. Choose a nonplanar 4-gon with vertices u_1, u_2, u_3, u_4 on X, cyclically ordered on this circle. Let h be a half-space whose boundary ∂h passes through the midpoints of u_1u_2, u_2u_3, u_3u_4, and u_4u_1, and such that h does not contain u_1. Then h contains u_2 and u_4 and not u_1 and u_3; thus $h \cap X$ is separated and X cannot be 0-tight.

(ii) Let X be a 0-tight 2-sphere in E^N. First let $N = 3$. Any topset of X is a point, a line segment, or an E_2-topset $Y = X_z$. We show that in the latter case Y has to be convex. Indeed, X contains the convex envelope $\partial\mathcal{H}Y \subset Y \subset X$. If Y is not convex then $\mathcal{H}Y \setminus Y$ contains at least one hole U as component and the open half-space $\mathring{h} = E^3 \setminus h_z$ intersects X in $X \setminus Y$, which has nonempty open pieces in X, separated by Y, also separated by the circle $\partial U \subset X = S^2$. This contradicts 0-tightness. So all topsets X_z of X are convex and the "convex" surface $\partial\mathcal{H}X$ is contained in the 2-sphere X. Then X is equal to this convex surface $\partial\mathcal{H}X$.

Next suppose X is a 0-tight substantial 2-sphere in E^N for $N > 3$. Let Y be as before a nonconvex E_2-top*set. Then $\partial\mathcal{H}Y \subset Y \subset E^2 \subset E^N$. Choose x_1 and x_2 in $X \setminus Y \subset X \setminus \partial\mathcal{H}Y$ in different components of $X \setminus Y$. There exists a half-space h such that $x_1, x_2 \in h$, but $h \cap Y = \varnothing$. Then $h \cap X = h \cap (X \setminus Y)$ is separated, contradicting 0-tightness of X. So all E^2-top*sets are convex. If Y is an E^3-top*set, then all topsets of Y are convex and the 2-sphere $\partial\mathcal{H}Y \subset Y \subset X$ must be equal to X. It cannot be substantial in E^N, for $N > 3$. So there is no E_3-top*set. Let k be either N or the smallest number $k < N$ with $k > 3$ for which there is an E_k-top*set Y. Then all topsets of X and of Y, are convex. So $\partial\mathcal{H}X \subset X$ and $\partial\mathcal{H}Y \subset Y \subset X$. But the dimension of $\partial\mathcal{H}Y$ is $k - 1 > 2$. This is absurd for dimension reasons. The desired result follows.

(iii) A 0-tight 2-disc in E^2 is a convex disc by Theorem 1.2. We can therefore assume the 0-tight disc X embedded in E^N, where $N \geq 3$. First let $N = 3$. A 0-tight nonconvex top*set X_z is necessarily an E_2-topset X_z, obtained from the convex disc $\mathcal{H}X_z$ by deleting one or more open sets U. If U is one of them and ∂U is not the boundary ∂X of the disc X, then $X \setminus X_z = X \setminus h_z$ contains at least two points x_1 and x_2, which are separated by ∂U. As $X \setminus h_z = (E \setminus h_z) \cap X$ is then separated and $E \setminus h_z$ is an open half-space, this contradicts 0-tightness. There is only *one* boundary ∂X for X, so that only one topset of X is nonconvex and it has only one convex hole U. This is the conclusion of the theorem for $N = 3$.