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The Foundations of Mathematics in The Theory of Sets

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

4 W. Miller, Jr. *Symmetry and separation of variables*
6 H. Minc *Permanents*
11 W. B. Jones and W. J. Thron *Continued fractions*
12 N. F. G. Martin and J. W. England *Mathematical theory of entropy*
18 H. O. Fattorini *The Cauchy problem*
19 G. G. Lorentz, K. Jetter and S. D. Riemenschneider *Birkhoff interpolation*
21 W. T. Tutte *Graph theory*
22 J. R. Bastida *Field extensions and Galois theory*
23 J. R. Cannon *The one-dimensional heat equation*
25 A. Salomaa *Computation and automata*
26 N. White (ed.) *Theory of matroids*
27 N. H. Bingham, C. M. Goldie and J. L. Teugels *Regular variation*
28 P. P. Petrushev and V. A. Popov *Rational approximation of real functions*
29 N. White (ed.) *Combinatorial geometrics*
30 M. Pohst and H. Zassenhaus *Algorithmic algebraic number theory*
31 J. Aczel and J. Dhombres *Functional equations containing several variables*
32 M. Kuczma, B. Chozewski and R. Ger *Iterative functional equations*
33 R. V. Ambartzumian *Factorization calculus and geometric probability*
34 G. Gripenberg, S.-O. Londen and O. Staffans *Volterra integral and functional equations*
35 G. Gasper and M. Rahman *Basic hypergeometric series*
36 E. Torgersen *Comparison of statistical experiments*
37 A. Neumaier *Intervals methods for systems of equations*
38 N. Korneichuk *Exact constants in approximation theory*
39 R. A. Brualdi and H. J. Ryser *Combinatorial matrix theory*
40 N. White (ed.) *Matroid applications*
41 S. Sakai *Operator algebras in dynamical systems*
42 W. Hodges *Model theory*
43 H. Stahl and V. Totik *General orthogonal polynomials*
44 R. Schneider *Convex bodies*
45 G. Da Prato and J. Zabczyk *Stochastic equations in infinite dimensions*
46 A. Bjorner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler *Oriented matroids*
47 E. A. Edgar and L. Sucheston *Stopping times and directed processes*
48 C. Sims *Computation with finitely presented groups*
49 T. Palmer *Banach algebras and the general theory of *-algebras*
50 F. Borceux *Handbook of categorical algebra I*
51 F. Borceux *Handbook of categorical algebra II*
52 F. Borceux *Handbook of categorical algebra III*
54 A. Katok and B. Hassleblatt *Introduction to the modern theory of dynamical systems*
55 V. N. Sachkov *Combinatorial methods in discrete mathematics*
56 V. N. Sachkov *Probabilistic methods in discrete mathematics*
57 P. M. Cohn *Skew Fields*
58 Richard J. Gardner *Geometric tomography*
59 George A. Baker, Jr. and Peter Graves-Morris *Padé approximants*
60 Jan Krajicek *Bounded arithmetic, propositional logic, and complex theory*
61 H. Gromer *Geometric applications of Fourier series and spherical harmonics*
62 H. O. Fattorini *Infinite dimensional optimization and control theory*
63 A. C. Thompson *Minkowski geometry*
64 R. B. Bapat and T. E. S. Raghavan *Nonnegative matrices and applications*
65 K. Engel *Sperner theory*
66 D. Cvetkovic, P. Rowlinson and S. Simic *Eigenspaces of graphs*
67 F. Bergeron, G. Labelle and P. Leroux *Combinatorial species and tree-like structures*
68 R. Goodman and N. Wallach *Representations of the classical groups*
69 T. Beth, D. Jungnickel and H. Lenz *Design Theory volume I 2 ed.*
70 A. Pietsch and J. Wenzel *Orthonormal systems and Banach space geometry*
71 George E. Andrews, Richard Askey and Ranjan Roy *Special Functions*
72 R. Ticciati *Quantum field theory for mathematicians*
76 A. A. Ivanov *Geometry of sporadic groups I*
78 T. Beth, D. Jungnickel and H. Lenz *Design Theory volume II 2 ed.*
80 O. Stormark *Lie's Structural Approach to PDE Systems*

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To Mary Penn Mayberry and Anita Kay Bartlett Mayberry

Contents

Preface

Dancing Master: *All the troubles of mankind, all the miseries which make up history, all the blunders of statesmen, all the failures of great captains – all these come from not knowing how to dance.*
Le Bourgeois Gentilhomme, Act 1, Scene 2

The importance of set-theoretical foundations

The discovery of the so-called "paradoxes" of set theory at the beginning of the twentieth century precipitated a profound crisis in the foundations of mathematics. This crisis was the more serious in that the then new developments in the theory of sets had allowed mathematicians to solve earlier difficulties that had arisen in the logical foundations of geometry and analysis. More than that, the new, set-theoretical approach to analysis had completely transformed that subject, allowing mathematicians to make rapid progress in areas previously inaccessible (in the theory of measure and integration, for example).

All of these advances seemed to be placed in jeopardy by the discovery of the paradoxes. Indeed, it seemed that mathematics itself was under threat. Clearly a retreat to the *status quo ante* was not an option, for serious difficulties once seen cannot just be ignored. But without secure foundations – clear concepts that can be employed without prior definition and true principles that can be asserted without prior justification – the very notion of *proof* is undermined. And, of course, it is the demand for rigorous proof that, since the time of the Greeks, has distinguished mathematics from all of the other sciences.

This crisis profoundly affected some mathematicians' attitudes to their subject. Von Neumann, for example, confessed in a brief autobiographical essay that the existence of the paradoxes of set theory cast a blight on his entire career, and that whenever he encountered technical difficulties in his research he could not suppress the discouraging thought that the problems in the foundations of mathematics doomed the whole mathematical enterprise to failure, in any case.

Mathematics, however, has passed through this crisis, and it is unlikely that a contemporary mathematician would suffer the doubts that von Neumann suffered. Indeed, mathematicians, in general, do not *worry* about foundational questions now, and many, perhaps most, of them are not even interested in such matters. It is surely natural to ask what is the cause of this complacency and whether it is justified.

Of course, every mathematician must master some of the facts about the foundations of his subject, if only to acquire the basic tools and techniques of his trade. But these facts, which are, essentially, just the elements of set theory, can be, and usually are, presented in a form which leaves the impression that they are just definitions or even mere notational conventions, so that their existential content is overlooked. What is more, the exposition of such foundational matters typically begins *in medias res*, so to speak, with the natural numbers and real numbers simply regarded as *given*, so that the beginner is not even aware that these things require proper mathematical definitions, and that those definitions must be shown to be both logically consistent and adequate to characterise the concepts being defined.

These fundamental number systems are nowadays defined using the axiomatic method. But there is a surprisingly widespread misunderstanding among mathematicians concerning the underlying logic of the axiomatic method. The result is that many of them regard the foundations of mathematics as just a branch of mathematical logic, and this encourages them to believe that the foundations of their subject can be safely left in the hands of expert colleagues. But formal mathematical logic itself rests on the same assumptions as do the other branches of mathematics: it, too, stands in need of foundations. Indeed, mathematical logicians are as prone to confusion over the foundations of the axiomatic method as their colleagues.

But this complacency about foundations does have a certain practical justification: modern mathematics does, indeed, rest on a solid and safe foundation, more solid and more safe than most mathematicians realise. Moreover, since mathematics is largely a technical, as opposed to philosophical, discipline, it is not unreasonable that mathematicians should, in the main, get on with the business of pursuing their technical specialities without worrying unduly about foundational questions. But that does not give them licence to pronounce upon matters on which they have not seriously reflected and are ignorant, or to assume that expertise in some special branch of their subject gives them special insight into its foundations.

However, even though it is not, strictly speaking, always necessary for mathematicians to acquire more than a basic knowledge of the foundations of their subject, surely it is desirable that they should do so. Surely the practitioners of a subject the very essence of which is proof and definition ought to be curious about the concepts and principles on which those activities rest.

Philosophers too have an important stake in these questions. Indeed, it is the fundamental role accorded to questions in the philosophy of mathematics that is the characterising feature of western philosophy, the feature that sharply distinguishes it from the other great philosophical traditions.

Problems relating to mathematics and its foundations are to be found everywhere in the writings of Plato and Aristotle, and every major modern philosopher has felt compelled to address them[1]. The subjects that traditionally constitute the central technical disciplines of philosophy – logic, epistemology, and metaphysics – cannot be studied in any depth without encountering problems in the foundations of mathematics. Indeed, the deepest and most difficult problems in those subjects often find their most perspicuous formulations when they are specialised to mathematics and its foundations. Even theology must look to the foundations of mathematics for the clearest and most profound study yet made of the nature of the infinite.

Unfortunately, the complacency, already alluded to, among mathematicians concerning the foundations of their subject has had a deleterious effect on philosophy. Deferring to their mathematical colleagues' technical competence, philosophers are sometimes not sufficiently critical of received opinions even when those opinions are patently absurd.

The mathematician who holds foolish philosophical opinions – about the nature of truth or of proof, for example – is protected from the consequences of his folly if he is prepared to conform to the customs and

[1] This is notoriously the case with Descartes, Leibniz, Kant, and, of course, Frege, who is the founder of the modern analytic school of philosophy; but it is no less true of Berkeley, Hume, and Schopenhauer. Among twentieth century philosophers, Husserl, Russell, and Wittgenstein come to mind.

mores of his professional tribe. But the philosopher who follows him in adopting those opinions does not have that advantage.

In any case, it is one thing to flirt with anarchist views if one lives in a settled, just, and well-policed society, but quite another if one is living in a society in which the institutions of law and justice threaten to collapse. Twice in the last two hundred years mathematicians have been threatened with anarchy – during the early nineteenth century crisis in the foundations of analysis and the early twentieth century crisis in the foundations of set theory – and in both of these crises some of the best mathematicians of the day turned their attention to re-establishing order.

The essential elements of the set-theoretical approach to mathematics were already in place by the early 1920s, and by the middle of the century the central branches of the subject – arithmetic, algebra, geometry, analysis, and logic – had all been recast in the new set-theoretical style. The result is that set theory and its methods now permeate the whole of mathematics, and the idea that the foundations of all of mathematics, including mathematical logic and the axiomatic method, now lie in the theory of sets is not so much a theory as it is a straightforward observation.

Of course that, on its own, doesn't mean that set theory is a *suitable* foundation, or that it doesn't require justification. But it does mean that any would-be reformer had better have something more substantial than a handful of new formalised axioms emblazoned on his banner. And he had better take it into account that even mathematical logic rests on set-theoretical foundations, and so is not available to him unless he is prepared to reform *its* foundations.

The point of view embodied in this book

My approach to set theory rests on one central idea, namely, that the modern notion of *set* is a refined and generalised version of the classical Greek notion of *number* (*arithmos*), the notion of number found in Aristotle and expounded in Book VII of Euclid's *Elements*. I arrived at this view of set theory more than twenty years ago when I first read *Greek Mathematical Thought and the Origin of Algebra* by the distinguished philosopher and scholar Jacob Klein.

Klein's aim was to explain the rise of modern algebra in the sixteenth and seventeenth centuries, and the profound change in the traditional concept of number that accompanied it. But it struck me then with the force of revelation that the later, nineteenth century revolution in the foundations of mathematics, rooted, as it was, in Cantor's new theory of transfinite numbers, was essentially a return to Greek arithmetic as Klein had described it, but in a new, non-Euclidean form.

As Klein points out, in Greek mathematics a number was defined to be a finite plurality composed of units, so what the Greeks called a number (*arithmos*) is not at all like what *we* call a number but more like what we call a set. It is having a finite size (cardinality) which makes a plurality a "number" in this ancient sense. But what is it for a plurality to have a finite size? *That* is the crucial question.

The Greeks had a clear answer: for them a definite quantity, whether continuous like a line segment, or discrete – a "number" in their sense – must satisfy the axiom that *the whole is greater than the part*[2]. We obtain the modern, Cantorian notion of set from the ancient notion of number by abandoning this axiom and acknowledging as finite, in the root and original sense of "finite"– "limited", "bounded", "determinate", "definite" – certain pluralities (most notably, the plurality composed of all natural numbers, suitably defined) which on the traditional view would have been deemed infinite.

By abandoning the Euclidean axiom that the whole is greater than the part, Cantor arrived at a new, *non-Euclidean arithmetic*, just as Gauss, Lobachevski, and Bolyai arrived at *non-Euclidean geometry* by abandoning Euclid's Axiom of Parallels. Cantor's innovation can thus be seen as part of a wider nineteenth century program of correcting and generalising Euclid.

Cantor's non-Euclideanism is much more important even than that of the geometers, for his new version of classical arithmetic that we call *set theory* serves as the foundation for the whole of modern mathematics, including geometry itself. The set-theoretical approach to mathematics is now taken by the overwhelming majority of mathematicians: it is embodied in the mathematical curricula of all the major universities and is reflected in the standards of exposition demanded by all the major professional journals.

Since the whole of mathematics rests upon the notion of set, this view of set theory entails that the whole of mathematics is contained in *arithmetic*, provided that we understand "arithmetic" in its original and historic sense, and adopt the Cantorian version of finiteness. In set theory, and the mathematics which it supports and sustains, we have

[2] This is Common Notion 5 in Book I of the *Elements*.

made real the seventeenth century dream of a *mathesis universalis*, in which it is possible to express the exact part of our thought[3].

But what are the practical consequences of this way of looking at set theory for mathematics and its foundations? They are, I am convinced, profound and far-reaching, both for orthodox set-theoretical foundations, and for the several dissenting and heterodox schools that go under various names – "constructivism", "intuitionism", "finitism", "ultra-intuitionism", etc. – but whose common theme is the rejection of the great revolution in mathematical practice that was effected by Cantor and his followers.

For orthodox foundations the principal benefit of looking at things in this way is that it enables us to see that the central principles – axioms – of set theory are really *finiteness principles* which, in effect, assert that certain multitudes (pluralities, classes, species) are finite in extent and *for that reason* form sets.

Taking finitude (in Cantor's new sense) to be the defining characteristic of sets, as the Greeks took it (in their sense) to be the defining characteristic of numbers (*arithmoi*), allows us to see why the conventionally accepted axioms for set theory – the Zermelo–Fraenkel axioms – are both natural and obvious, and why the unrestricted comprehension principle, which is often *claimed* as natural and obvious (though, unfortunately, self-contradictory), is neither.

This is a matter of considerable significance, for there is a widespread view that all existing axiomatisations of set theory are more or less *ad hoc* attempts to salvage as much of the "natural" unrestricted comprehension principle – the principle that the extension of any well-defined property is a set – as is consistent with avoiding outright self-contradiction[4]. On this view set theory is an unhappy compromise, a botched job at best.

Hence the widespread idea that set theory must be presented as an axiomatic theory, indeed, as an axiomatic theory *formalised* in first order mathematical logic. It is felt that the very formalisation itself somehow confers mathematical respectability on the theory formalised. But this is a serious confusion, based on a profound misunderstanding of the logical and, indeed, *ontological* presuppositions that underlie the axiomatic method, formal or informal.

The mathematician's "set" is the mathematical logician's "domain of discourse", so conventional ("classical") mathematical logic is, like every

[3] Perhaps we might more appropriately describe the theory as an *arithmetica universalis*, a *universal arithmetic* which encompasses the whole of mathematics.
[4] See Quine's *Set Theory and its Logic*, for example.

other branch of mathematics, based on set theory[5]. This means, among other things, that we cannot use the standard axiomatic method to establish the theory of sets, on pain of a circularity in our reasoning.

Moreover, on the arithmetical conception of set the totality of all sets, since it is easily seen not to be a set, is not a conventional domain of discourse either. Hence quantification over that non-conventional domain (which is *absolutely infinite* in Cantor's terminology) cannot simply be *assumed* to conform to the conventional, "classical" laws.

As Brouwer repeatedly emphasised, since classical logic is the logic of the finite, the logic of infinite domains must employ different laws. And, of course, in the present context "finite domain" simply means "set". The consequences of this view for the *global* logic of set theory are discussed at length in Section 3.5 and Section 7.2.

But what are the consequences of this *arithmetical* conception of set for those who reject Cantor's innovations – the intuitionists, finitists, constructivists, etc., of the various schools?

Klein's profound scholarship is very much to the point here. For the one thing on which all these schools agree is the central importance of the system of natural numbers as the basic *datum* of mathematics. But Klein shows us that, on the contrary, the natural numbers are a recent invention: the oldest mathematical concept we have is that of *finite plurality* – the Greek notion of *arithmos*. This is so important a matter that I have devoted an entire chapter (Chapter 2) to its dicussion.

When the natural number system is taken as a primary *datum*, something simply "given", it is natural to see the principles of proof by mathematical induction and definition by recursion along that system as "given" as well. We gain our knowledge of these numbers when we learn to count them out and to calculate with them, so we are led to see these *processes* of counting out and calculating as *constitutive* of the very notion of natural number. The natural numbers are thus seen as what we arrive at in the *process* of counting out: $0, 1, 2, \ldots$, where the dots of ellipsis, "…", are seen as somehow self-explanatory – after all, we all know how to continue the count no matter how we have taken it. But those dots of ellipsis contain the whole mystery of the notion of natural number!

If, however, we see the notion of natural number as a secondary

[5] Thus set theory stands the "logicist" view of Frege and Russell on its head: arithmetic isn't a branch of logic, logic is a branch of arithmetic, the non-Euclidean arithmetic of Cantor that we call set theory.

growth on the more fundamental notion of *arithmos* – finite plurality, in the original Greek sense of "finite" – then the principles of proof by induction and definition by recursion are no longer just "given" as part of the raw data, so to speak, but must be established from more fundamental, set-theoretical principles.

Nor are the *operations* of counting out or calculating to be taken as primary data: they too must be analysed in terms of more fundamental notions. We are thus led to reject the *operationalism* that all the anti-Cantorian schools share.

For us moderns numbers take their being from what we can *do* with them, namely count and calculate; but Greek "numbers" (*arithmoi*) were objects in their own right with simple, intelligible natures. Our natural numbers are things that we can (in principle) *construct* (by counting out to them); Greek numbers were simply "there", so to speak, and it would not have occurred to them that their numbers had to be "constructed" one unit at a time[6].

I am convinced that this operationalist conception of natural number is the central fallacy that underlies *all* our thinking about the foundations of mathematics. It is not confined to heretics, but is shared by the orthodox Cantorian majority. This *operationalist fallacy* consists in the assumption that the mere *description* of the natural number system as "what we obtain from zero by successive additions of one" suffices *on its own* to define the natural number system as a unique mathematical structure – the assumption that the operationalist description of the natural numbers is itself what provides us with a *guarantee* that the system of natural numbers has a unique, fixed structure.

Let me not be mistaken here: the existence of a unique (up to isomorphism) natural number system is a *theorem* of orthodox, Cantorian mathematics. The fallacy referred to thus does *not* consist in supposing that *there is* a unique system of natural numbers, but rather in supposing that the existence of this system, and its uniqueness, are immediately given and do not need to be *proved*. And if we abandon Cantorian orthodoxy we thereby abandon the means with which to prove these things.

[6] Oswald Spengler, who thought that the mathematics of a civilization held a clue to its innermost nature, contrasted the *Apollonian* culture of classical Greece, which was static and contemplative, with the *Faustian* culture of modern Europe, which is dynamic and active. Whatever the virtues of his general thesis, he seems to have got it right about the mathematics. The "operationalism" to which I refer here seems to be quintessentially Faustian in his sense, which perhaps explains its grip on our imaginations.

But if we acknowledge that the natural numbers are not *given* to us, the alternative, if we decide to reject Cantor's radical new version of finitude, is to return to arithmetic as practiced by the mathematicians of classical Greece, but equipped now with the more powerful and more subtle techniques of modern set theory. If we should decide to do this we should be going back to the very roots of our mathematical culture, back before Euclid and Eudoxus to its earliest Pythagorean origins. We should have to rethink our approach to geometry and the Calculus. It is a daunting prospect, though an exciting one.

The resulting theory, which I call *Euclidean set theory* by way of contrast with *Cantorian set theory*, the modern orthodoxy, is very like its Cantorian counterpart, except that Cantor's assumption that the species of natural numbers forms a set is replaced by the traditional Euclidean assumption that every set is strictly larger than any of its proper subsets.

This theory, not surprisingly, constitutes a radical departure from Cantorian orthodoxy. But it stands in even sharper contrast to the various operationalist theories which have been put forward as alternatives to that orthodoxy. So far from taking the natural numbers as given, Euclidean set theory forces us to take seriously the possibility that there is no unique natural number system, and that the various ways of attempting to form such a system lead to "natural number systems" of differing lengths.

But *should* we abandon Cantorian orthodoxy? There is obviously a *prima facie* case against the Cantorian account of finiteness, and, indeed, that case was made by some of his contemporaries. But against that there is the experience of more than one hundred years during which Cantor's ideas have been the engine driving a quite astonishing increase in the subtlety, power, and scope of mathematics.

Perhaps I should come clean with the reader and admit that I am attracted to the anti-Cantorian position. I put it no stronger than that because the issue is by no means clear-cut, and we do not yet know enough to be sure that the Cantorian conception of finiteness should be rejected.

Indeed, it seems to me that the common failing of all the advocates of the various alternatives to Cantorian orthodoxy is that they fail to appreciate how simple, coherent, and plausible are the foundational ideas that underlie it. These enthusiasts rush forward with their proposed cures without having first carried out a proper diagnosis to determine the nature of the disease, or even whether there *is* a disease that requires their ministrations.

Accordingly, I shall devote much of my attention to a careful, sympathetic, and detailed treatment of the Cantorian version of the theory. This is of interest in its own right, for this is the theory on which all of current mathematics rests. But it is also essential for those who are dissatisfied (or who fancy themselves dissatisfied) with the current orthodoxy, to discover what principles that orthodoxy really rests on, and to determine exactly where its strengths and weaknesses lie.

I have divided my exposition into four parts. *Part One* deals with the criteria which any attempt to provide foundations for mathematics must meet, and with the significance of the Greek approach to arithmetic for modern foundations.

Part Two is an exposition of the elements of set theory: the basic concepts of set theory, which neither require, nor admit of, definition, but in terms of which all other mathematical concepts are defined; and the basic truths of set theory, which neither require, nor admit of, proof, but which serve as the ultimate assumptions on which all mathematical proofs ultimately rest. The theory presented in *Part Two* is common to both the Cantorian and the Euclidean versions of set theory.

Part Three is an exposition of the Cantorian version of the theory and *Part Four* of the Euclidean. I have also included an appendix which deals with logical technicalities.

This, then, is the point of view embodied in this book: all of mathematics is rooted in *arithmetic*, for the central concept in mathematics is the concept of a plurality limited, or bounded, or determinate, or definite – in short, *finite* – in size, the ancient concept of *number* (*arithmos*).

From this it follows that there are really only two central tasks for the foundations of mathematics:

1. To determine what it is to be *finite*, that is to say, to discover what basic principles apply to finite pluralities *by virtue of their being finite*.
2. To determine what logical principles should govern our reasoning about *infinite* and *indefinite* pluralities, pluralities that are *not* finite in size.

On this analysis, all disputes about the proper foundations for mathematics arise out of differing solutions to these two central problems.

Such a way of looking at things is not easily to assimilate to any of the well-known "isms" that have served to describe the various approaches to the study of mathematical foundations in the twentieth century. But to my mind it has a certain attractive simplicity. Moreover, it is rooted in the history of mathematics and, indeed, takes as its starting point the oldest mathematical concept that we possess.

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Moreover, it was he who called my attention to the importance of the Ackermann simply infinite system treated in Section 10.6, and who, with the help of a suggestion by Robin Gandy, laid down a definition of that system in the special case of pure sets. Without his work this would have been a very different book.

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