Part One
Preliminaries

It is the mark of an educated man to look for precision in each kind of enquiry just to the extent that the nature of the subject allows.

Aristotle

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1

The Idea of Foundations for Mathematics

1.1 Why mathematics needs foundations

Mathematics differs from all the other sciences in requiring that its propositions be proved. Certainly no one will deny that proof is the goal of mathematics, even though there may be disagreement over whether, or to what extent, that goal is achieved. But you cannot prove a proposition unless the concepts employed in formulating it are clear and unambiguous, and this means that the concepts used in a proof either must be basic concepts that can be grasped directly and can be seen immediately to be clear and unambiguous, or must be rigorously *defined* in terms of such basic concepts. Mathematics, therefore, since it is about proof is also about definition.

Now definition and proof are both species of the genus *explanation*: to define something is to explain *what* it is; to prove something is to explain *why* it is true. All scholars and scientists, of course, deal in explanation. But mathematicians are unique in that they intend their explanations to be complete and final: that must be their aim and ideal, even if they fail to realise it in full measure. From these simple observations many consequences flow.

Perhaps the most important of them concerns the mathematician's claims to truth. Because he deals in proof, those claims must be absolute and unqualified. Whether they are justified, either in general, or in particular cases, is, of course, quite another matter: but that they are, in fact, made cannot be denied without stripping the word "proof" of all meaning. To claim to have proved something is to claim, among other things, that it is true, that its truth is an objective fact, and that its being so is independent of all authority and of our wishes, customs, habits, and interests. Where there are no truth and falsehood, objectively determined, there can be no proof; and where there is no proof there can be no mathematics.

No doubt all of this is at odds with the *Zeitgeist*[1]: it would seem that we must come to terms with the fact that when there is disagreement about a genuine mathematical proposition, someone must be right and someone must be wrong. But the requirement that we must lay unqualified claims to truth in mathematics is quite compatible with our maintaining a prudent and healthy scepticism about such claims: what it rules out is dogmatic or theoretical scepticism.

You may, as a mathematician, reasonably doubt that such and such a theorem is true, or that such and such a proof is valid: indeed, there are many occasions on which it is your professional duty to do this, even to the point of struggling to maintain doubt that is crumbling under the pressure of argument: for it is precisely when you begin to settle into a conviction that you are most liable to be taken in by a specious but plausible line of reasoning. When your business is judging proofs you must become a kind of professional sceptic. But scepticism, properly understood, is an attitude of mind, not a theory, and you cannot systematically maintain that there is no such thing as a true proposition or a valid argument and remain a mathematician.

A proof, to be genuine, must still all reasonable doubts as to the truth of the proposition proved. But the doubts to be stilled are those that pertain to that proposition: a proof need not, indeed cannot, address general sceptical doubts. Anyone who proposes to pass judgement on the validity of an intended proof must address his attention to the propositions and inferences contained in the argument actually presented. It won't do to object to a particular argument on the ground that all argument is suspect. The fact, for example, that people often make mistakes in calculating sums does not provide grounds for concluding that any particular calculation is incorrect, or even uncertain: each must be judged separately, on its own merits.

In the final analysis, there are only two grounds upon which you may reasonably call the efficacy of a purported proof into question: you may dispute the presuppositions upon which the argument rests, or you may dispute the validity of one or more of the inferences by means of which the argument advances to its conclusion. If, after careful, and perhaps prolonged, reflection, you cannot raise an objection to an argument on

[1] Cantor complained of the "Pyrrhonic and Academic scepticism" that prevailed in his day. *Plus ça change* ...

either of these two grounds, then you should accept it as valid and its conclusion as true.

Here we must include among the presuppositions of a proof not only the truth of the propositions that are taken as unproved starting points of the argument, but also the clarity, unambiguity, and unequivocality of the concepts in which the propositions employed in the argument are couched.

Of course in practice, actual proofs start from previously established theorems and employ previously defined concepts. But if we persist in our analysis of a proof, always insisting that, where possible, assertions should be justified and concepts defined, we shall eventually reach the ultimate presuppositions of the proof: the propositions that must be accepted as true without further argument and the concepts that must be understood without further definition. Of course when I say that these things *must* be accepted without proof or understood without definition I mean that they must be so accepted and so understood if the given proof is to be judged valid and its conclusion true.

If we were to carry out such a complete analysis on all mathematical proofs, the totality of ultimate presuppositions we should then arrive at would obviously constitute the foundations upon which mathematics rests. Naturally, I'm not planning to embark on the enterprise of analysing actual proofs to discover those foundations. My point here is rather that solely in virtue of the fact that mathematics is about proof and definition it must of necessity *have* foundations, ultimate presuppositions – unproved assertions and undefined concepts – upon which its proofs and definitions rest.

Of course that observation is compatible with there being a motley of disparate principles and concepts underlying the various branches of the subject, with no overarching ideas that impose unity on the whole. The question thus arises whether it is possible to discover a small number of clear basic concepts and true first principles from which the whole of mathematics can be systematically developed: that is, I suspect, what most mathematicians have in mind when they speak of providing foundations for mathematics.

From the very beginnings of the subject, that is to say, from the time when proof became central in mathematics, mathematicians and philosophers have been aware of the need to provide for foundations in the ideal and general sense just described. But there are particular, and pressing, practical reasons why present day mathematics needs foundations in this sense. Mathematics today is, for mathematicians, radically different from what it was in the relatively recent past, say one hundred and fifty years ago, and, indeed, come to that, from what it is now for professional *users* of mathematics, such as physicists, engineers, and economists. The difference lies in the greatly enhanced role that definition now plays. Present day mathematics deals with rigorously defined mathematical structures: groups, rings, topological spaces, manifolds, categories, etc. Traditional mathematics, on the other hand, was based on geometrical and kinematical intuition. Its objects were idealised shapes and motions. They could be imagined – pictured in the mind's eye – but they could not be rigorously defined.

Now it is precisely in our possession of powerful and general methods of rigorous definition that we are unquestionably superior to our mathematical predecessors. However, this superiority does not consist primarily in our basic definitions being more certain or more secure – although, indeed, they are more certain and secure, as are the proofs that employ them – but rather in the fact that they can be generalised and modified to apply in circumstances widely remote from those in which they were originally conceived.

There is a certain irony here. For although the earliest pioneers of modern rigour – Weierstrass for example – set out in search of safer, more certain methods of definition and argument by cutting mathematics free of its former *logical* dependence on geometrical and kinematical intuitions, they have, paradoxically, enormously enlarged the domain in which those intuitions can be applied.

When we give a rigorous "analytic" (i.e. non-geometrical, non-kinematical) definition of "limit" or "derivative" we do, undoubtedly, attain a greater certainty in our proofs. But, what is just as important, we can *generalise* a rigorous, analytic definition, while a definition based on geometrical or kinematical intuition remains tied to what we can actually visualise. By purging our definitions of their *logical* dependence on geometrical and kinematical intuition, we clear the way for transferring our insights based on that intuition to "spaces", for example, infinite dimensional ones, in which intuition, in the Kantian sense of sensual intuition – images in the mind's eye – is impossible. The mathematicians of the nineteenth century noticed that by a novel use of definition they could convert problems in geometry into problems in algebra and set theory, which are more amenable to rigorous treatment[2]. What they didn't

[2] Descartes saw that problems in geometry could be converted into problems in algebra. But his algebra, the algebra of real numbers, rested logically on geometrical conceptions.

foresee – how could they have foreseen it? – was the enormous increase in the scope of mathematics that these new methods made possible. By banishing "intuitive" ("*anschaulich*") geometry from the logical foundations of mathematics, they inadvertently, and quite unintentionally, gave that geometry a new lease of life.

But it was the technique of axiomatic definition that made the transition from traditional to modern mathematics possible. Naively, an axiomatic definition defines a *kind* or *species* of mathematical structure (e.g. *groups, rings, topological spaces, categories,* etc.) by laying down conditions or *axioms* that a structure must satisfy in order to be of that kind. Axiomatic definition is the principal tool employed in purging the foundations of mathematics of all *logical* dependence on geometrical and kinematical intuition. It follows that if we wish to understand *how* geometry has disappeared from the logical foundations of mathematics, we must understand the logical underpinnings of axiomatic definition. To understand those underpinnings is to understand how set theory provides the foundations for all mathematics.

Here we come to the central reason why modern mathematics especially stands in need of a careful examination and exposition of its foundations. For there is widespread confusion concerning the very nature of the modern axiomatic method and, in particular, concerning the essential and ineliminable role set theory plays in that method[3]. I shall discuss this critical issue later in some detail[4]. But for now, suffice it to say that the *logical* dependence of axiomatics on the set-theoretical concept of mathematical structure requires that set theory already be in place before an account of the axiomatic method, understood in the modern sense of axiomatic *definition*, can be given. It follows necessarily, therefore, that *we cannot use the modern axiomatic method to establish the theory of sets.* We cannot, in particular, simply employ the machinery of modern logic, modern *mathematical* logic, in establishing the theory of sets.

There is, to be sure, such a thing as "axiomatic set theory"; but although this theory is of central importance for the study of the foundations of mathematics, *it is a matter of logic* that it cannot itself, as an axiomatic theory in the modern sense, serve as a foundation for mathematics. Set theory, as a foundational theory, is, indeed, an axiomatic

The novelty introduced by later mathematicians was to base the algebra of real numbers on set theory, using the technique of axiomatic definition.

[3] I have discussed this matter at some length in my article "What is required of a foundation for mathematics?" to which I refer the interested reader.

[4] Chapter 6, especially Sections 6.2, 6.3 and 6.4.

theory, but in the original sense of "axiomatic" that applies to traditional Euclidean geometry as traditionally understood. The axioms of set theory are not conditions that single out a class of interpretations, as are, for example, Hilbert's axioms for geometry. On the contrary, they are fundamental truths expressed in a language whose fundamental vocabulary must be understood *prior* to the laying down of the axioms. That, in any case, must be the view taken of those axioms by anyone who embarks on the enterprise of expounding the set-theoretical foundations of mathematics. Whether, or to what extent, any such enterprise is successful, whether, or to what extent, the axioms can legitimately be regarded in this manner, is, of course, a matter for judgement. But it will be a central part of my task to show that they can be so regarded.

1.2 What the foundations of mathematics consist in

As I have just explained, the foundations of mathematics comprise those ideas, principles, and techniques that make rigorous proof and rigorous definition possible. To expound those foundations systematically, one must provide three things: an account of the *elements* of mathematics, an account of its *principles*, and an account of its *methods*.

The *elements* of mathematics are its basic notions: the fundamental *concepts* of mathematics, the *objects* that fall under those concepts, and the fundamental *relations* and *operations* that apply to them. These basic notions are those that neither require, nor admit of, proper mathematical definition, but in terms of which all other mathematical notions are ultimately defined. Insofar as these basic notions of mathematics are clear and unambiguous, the basic propositions of mathematics, which employ them, will also be clear and unambiguous. In particular, those propositions will have objectively determined truth values: the truth or falsity of such a proposition will be a question of objective fact, not a mere matter of convention or of agreement among experts.

The *principles* of mathematics are its *axioms*, properly so called. They are fundamental propositions that, although true, neither require, nor admit of, proof; and they constitute the ultimate and primary assumptions upon which all mathematical argument finally rests. There is no sense in which the axioms can be construed as giving or determining the meaning of the vocabulary in which they are couched. On the contrary, the meanings of the various items of vocabulary must be given, in advance of the laying down of the axioms, in terms of the elements of the theory, antecedently understood.

The *methods* of mathematics are to be given by laying down the canons of definition and of argument that govern the introduction of new concepts and the construction of proofs. This amounts to specifying the *logic of mathematics*, which we must take care to distinguish from *mathematical logic*: mathematical logic is a particular branch of mathematics, whereas the logic of mathematics governs all mathematical reasoning, including reasoning about the formal languages of mathematical logic and their interpretations. The logic of mathematics cannot be purely formal, since the propositions to which it applies have fixed meanings and the proofs it sanctions are meaningful arguments, not just formal assemblages of signs.

Here it must be said that the need to include an explicit account of logical method is a peculiarity of modern mathematics. Under the Euclidean dispensation, before the advent of set theory as a foundational theory, and when definition played a much more modest role in mathematics, one could, or, in any event, one did, take one's logic more for granted. But with the rise of modern mathematics, in which definition has moved to the centre of the stage, and where mathematicians have gone beyond even Euclid in their quest for accuracy and rigour, it has become necessary to include logical methods among the foundations of the subject. In fact, the central problem here is to explain the logical principles that underlie the modern axiomatic method. This will raise questions of the logic of generality, of the *global* logic of mathematics, that are especially important, and especially delicate, as we shall see[5].

A systematic presentation of the foundations of mathematics thus consists in a presentation of its elements, its principles, and its logical methods. In presenting these things we must strive for *simplicity, clarity, brevity*, and *unity*. These are not mere empty slogans. The requirements for *simplicity* and *clarity* mean, for example, that we cannot take sophisticated mathematical concepts, such as the concept of a category or the concept of a topos, as *foundational* concepts, and that we cannot incorporate "deep" and controversial philosophical theories in our mathematical foundations. Otherwise no one will understand our definitions and no one will be convinced by our proofs.

The ideal of *brevity*, surely, speaks for itself. *Unity* has always been a central goal: unity in principles, unity in logical technique, unity in standards of rigour. With the stupendous expansion that has taken place in mathematics since the middle of the nineteenth century the

[5] I shall discuss this point in Sections 3.4 and 3.5.

need to strive for unity in foundations is even more pressing than ever: mathematics must not be allowed to degenerate into a motley of mutually incomprehensible subdisciplines.

This, then, is what an exposition of the foundations of mathematics must contain, and these are the ideals that must inform such an exposition. But the task of *expounding* the foundations of mathematics must be kept separate from the task of *justifying* them: this is required by the logical role that those foundations are called upon to play. A little reflection will disclose, indeed it is obvious, that there can be no question of a *rigorous* justification of proposed foundations: if such a justification were given, then the elements, principles, and logical methods presupposed by that justification would themselves become the foundations of mathematics, properly so called.

Thus the clarity of basic concepts (if they really are basic) and the truth of first principles (if they really are *first* principles) cannot be established by rigorous argument of the sort that mathematicians are accustomed to. Insofar as these things are evident they must be *self*-evident. But that is not to say they are beyond justification; it is only to say that the justification must proceed by persuasion rather than by demonstration: it must be dialectical rather than apodeictic.

In any case, self-evidence, unlike truth, admits of degrees, and, as we shall see, the set-theoretical axioms that sustain modern mathematics are self-evident in differing degrees. One of them – indeed, the most important of them, namely Cantor's Axiom, the so-called Axiom of Infinity – has scarcely any claim to self-evidence at all, and it is one of my principal aims to investigate the possibility, and the consequences, of rejecting it. But what is essential here is this: when we lay down a proposition *as an axiom* what we are thereby claiming directly is that it is *true*; the claim that it is self-evident is, at most, only implicit, and, in any case, is *logically* irrelevant.

1.3 What the foundations of mathematics need not include

It is obvious to anyone who teaches mathematics that means must be devised for presenting its foundations simply, yet rigorously and thoroughly, to apprentice mathematicians: they must be told about sets, about ordered pairs and Cartesian products, about functions and relations; they must be made to grasp the idea of mathematical structure, and of a morphology-preserving map between such structures; more generally, they must be taught the techniques of rigorous proof and rigorous