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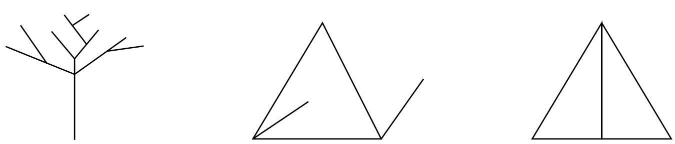
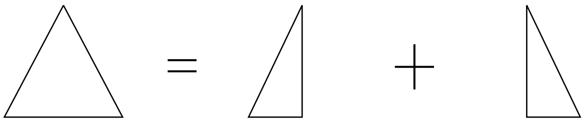
Introduction

Algebraic methods have often been applied to the study of geometry. The algebraic reformulation of euclidean geometry as ‘coordinate geometry’ by Descartes was peculiarly successful in that algebra and geometry exactly mirrored each other. The objects studied in this book – graphs, surfaces and higher dimensional ‘complexes’ – are much more complicated than the lines and circles of euclidean geometry and it is not to be expected that any reasonable algebraic machinery can hope to describe all their features. We must expect many distinct geometrical situations to be described by the same piece of algebra. That is not necessarily a disadvantage, for not all features are equally interesting and it can be useful to have a systematic way of ignoring some of the less interesting ones.

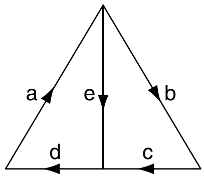
Throughout the book we shall be concerned with subsets X of a euclidean space (such as ordinary space $\mathbb{R}^3$) and shall attempt to clarify the structure of the sets X by looking at what are called *cycles* on them. For the present let us consider one-dimensional cycles, or 1-cycles, which have their origin in the idea of simple closed curve (that is, a closed curve without self-intersections). Thus consider the three graphs drawn below. It is fairly evident that no simple closed curves at all can be drawn on the left-hand graph.



On the middle graph a simple closed curve can be drawn going once round the triangle, while on the right-hand graph there are three triangles, two right-angled and one equilateral, which can be circumnavigated. The first hint of algebra comes when we try to combine two cycles (in this case two simple closed curves) together. We want to say something like



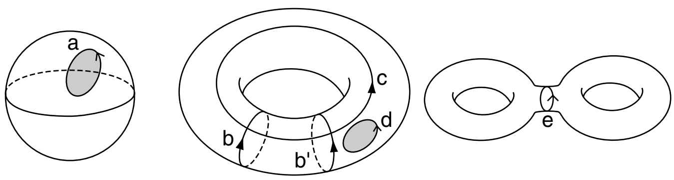
The way in which we make the vertical edge cancel out is by giving orientations (directions) to the edges of the triangles and taking all cycles to be, say, clockwise. Then we have



$$a + b + c + d = (a + e + d) + (b + c - e)$$

where the $-e$ occurs because e is going the wrong way for the right-hand cycle. The sum now looks like a calculation in an abelian group, and indeed in Chapter 1 we make the set of 1-cycles into an abelian group which measures the 'number of holes' in the graph, or the 'number of independent loops' on the graph. This number is respectively 0, 1 and 2 for the three graphs above, and the corresponding groups are 0, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$. The amount of group theory needed to read Chapter 1 is minimal, barely more than the definitions of abelian group and homomorphism.

When we come to consider 1-cycles on higher-dimensional objects the situation becomes more interesting. The diagrams



represent a spherical surface, a torus (or 'inner tube') and a double torus or pretzel, and a, b, b', c, d, e are 1-cycles. Now a and d are each the boundary of a region – such a region is shaded, but it would also be possible to use the 'complementary' region occupying the remainder of the surface. On the other hand b, b' and c do not bound regions. The fact that the simple closed curve b fails to bound a region reveals the presence of a hole in the torus (where the air goes in an inner tube): 'b goes round the hole'. Similarly c reveals the presence of another hole (where the axle of the wheel goes). Since b and c are one-dimensional we could call the holes one-dimensional too.

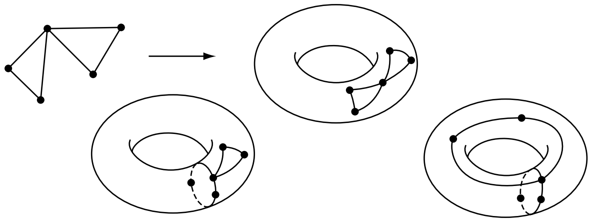
In order to use cycles for gathering geometric information we shall declare those which, like a and d, bound a region, to be 'zero' – homologous to zero is the official phrase, denoted ~ 0. Now b and b' between them bound a region; once the idea of orientation is made precise this implies that $b - b' \sim 0$ (rather than $b + b' \sim 0$). Rearranging, we have $b \sim b'$, which gives formal expression to the fact that b and b' go round the same hole in the torus. There is a technique available for making the bounding cycles zero: they form a subgroup (the *boundary* subgroup) of the group of cycles and what is required is the quotient group $\{\text{cycles}\}/\{\text{boundaries}\}$. This quotient group, when the cycles are 1-cycles, is called the *first homology group*. The complexity ('number of free generators') of this group measures, roughly speaking, the number of one-dimensional holes the object has. The sphere has none and the torus has two; their first homology groups are respectively 0 and $\mathbb{Z} \oplus \mathbb{Z}$.

Notice in passing that a and d can each be shrunk to a point on its respective surface whereas e, on the double torus, cannot. Nevertheless e does bound a region (namely one half of the surface) so $e \sim 0$.

The whole of, for example, the spherical surface can be considered as a two-dimensional cycle, or 2-cycle, since it is 'closed'; certainly this cycle is not the boundary of anything since there is nothing three-dimensional *in the surface* for it to bound. If on the other hand we replace the sphere by the solid ball obtained by filling in the inside of the sphere, then the 2-cycle given by the sphere becomes a boundary: in the solid ball this 2-cycle is a boundary. As before we can form the homology group, this time the *second homology group*; for the sphere this is $\mathbb{Z}$ and for the solid ball it is 0.

Notice that if we consider a solid torus (like a solid tyre) then b and b' become homologous to zero, while c does not. In fact the first homology group changes from $\mathbb{Z} \oplus \mathbb{Z}$ to $\mathbb{Z}$.

It is important to realize that the homology groups of an object depend only on the object and not on the way in which it is placed in space. Thus if we cut the torus along b, tie it in a knot and then re-stick along b, the resulting 'knotted torus' has the same homology groups as the original one. Stepping down one dimension we could consider the same graph being placed in different ways in a surface, such as the torus in the diagrams below:

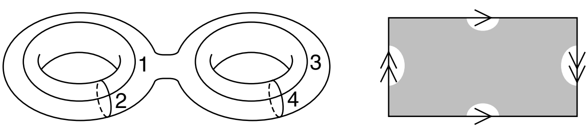


These three situations are quite different – for instance the number of regions into which the torus is divided is different in each case. We can study ‘embedding’ or ‘placement’ problems by means of *relative homology groups*, which take into account both graph and torus at the same time. The ordinary groups and the relative groups are introduced in Chapter 4.

Here are some of the problems which are studied in Chapter 9 by means of relative homology groups.

1. Given a graph in a surface, what can be said about the number of regions into which the graph divides the surface and about the nature of those regions? Much of Chapter 9 is concerned with this problem.

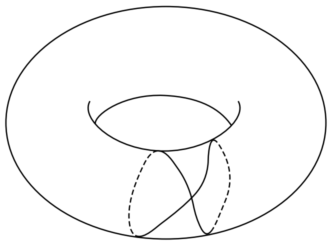
2. What is the ‘largest’ graph which can be embedded in a surface without separating it into two regions? Of course it is necessary to choose a good definition of largeness first; here is an alternative formulation of the question. How many simple closed cuts can be made on a surface before it falls into two pieces? For example on a double torus the answer is four – see the diagram on the left. For a ‘Klein bottle’, obtained from a plane rectangle by identifying



the sides in pairs as indicated on the diagram on the right (compare 2.2(4)), the answer is two. Can the reader find two suitable curves? The general answer is in 9.20. The Klein bottle cannot be constructed in three-dimensional space, and a proof that this is so is sketched in 9.18. It can be constructed in $\mathbb{R}^4$ (2.16(5)).

3. What can be said about a cycle given by a simple closed curve on, say, the torus? The simple closed curve labelled b in the middle diagram on page 2 winds round one way and the simple closed curve c winds round the other way. Can a simple closed curve wind round, say, twice the b-way and not go round

the *c*-way at all? Certainly the most obvious candidates are not *simple* closed curves:



The general answer is in 9.22.

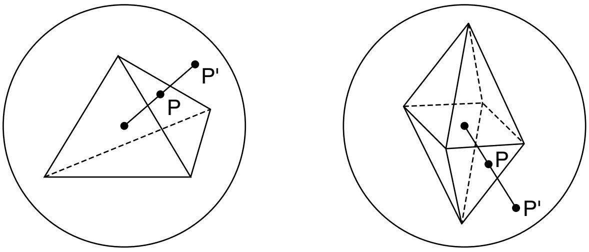
Curiously, although cycles were studied from the beginnings of algebraic topology in the writings of H. Poincaré at the end of the nineteenth century, no explicit mention was made of homology *groups* until much later, around the early 1930s. What were used instead were numerical invariants of the groups (the so-called rank and torsion coefficients; see A.26), which accounts for the lack of formal recognition given to many of the basic results. For example the exact homology sequence of a pair (Chapter 6) did not appear explicitly until 1941. The reformulation of homology theory in terms of abelian groups not only produced a substantial increase in clarity but made available powerful tools with which the scope of the theory could be broadened.†

This is an example – one of many – where modern algebraic formalism has come to the rescue of an attractive but elusive geometrical idea. Of course any rescue operation has its dangers; in this case that formalism should cease to be subservient to geometry and become an end in itself. 1 have tried very hard to avoid such a course and to keep constantly in mind that the goal of the book is the geometrical applications of Chapter 9.

The techniques developed in this book do not enable us to study 'curved surfaces' such as the torus or sphere directly. (The techniques of differential topology are more suited to such a direct study; for a beautiful introduction to that subject see the book of Milnor listed in the References.) Instead we 'triangulate' these surfaces, that is, divide them up into (for the moment curved) triangular regions in such a way that two regions which intersect do so either at a single vertex or along a single edge. Triangulating a surface reduces the calculation of homology groups to a finite procedure, which is explained in Chapter 4; it has the additional advantage that it is possible to give a complete prescription for constructing triangulations of surfaces. This is given in Chapter 2; as

† For more historical information see the books by Dieudonné, Lakatos *et al*., Kline, Eilenberg & Steenrod and Biggs, Lloyd & Wilson, and the article by Hilton (1988) listed in the References.

an example consider the diagrams below, showing a tetrahedron and an octahedron each inside a sphere. The hollow tetrahedron can be projected outwards ($P \to P'$) on to the sphere which surrounds it, thus giving a triangulation of the sphere. Likewise the hollow octahedron can be projected outwards ($P \to P'$) to give another triangulation. Indeed, the hollow tetrahedron and octa-



hedron can themselves be regarded as triangulated surfaces in which it so happens that the triangles are flat and the edges straight. These rectilinear triangulations are referred to as *simplicial complexes*; they are studied in considerable generality in Chapter 3. Their importance lies in the fact that any triangulated surface is homeomorphic to one in which the triangles are flat and the edges straight. (Two subsets X and Y of euclidean space are called *homeomorphic* if there exists a continuous, bijective map $f : X \to Y$ with continuous inverse f^{-1}. Such a map f is called a *homeomorphism*.) The maps $P \to P'$ above are both homeomorphisms. The kind of geometrical information which we are trying to capture – that is, information of a topological nature – is so basic that it is unaltered by homeomorphisms. Thus it is no restriction to concentrate entirely on rectilinear triangulations.

That is what we do in this book: all triangulations have flat triangles with straight edges. It should be noted, however, that often when drawing surfaces we shall draw them curved. This is for two reasons: the pictures are more recognizable and easier to draw; and they help to remind us that the triangulations are a tool to help us gather information, not an intrinsic part of the geometrical data.

The remarkable thing is that homology groups, in spite of being defined via triangulations, really do measure something intrinsic and geometrical. To be more precise: if X and Y are homeomorphic, then the homology groups of X are isomorphic to those of Y. This is true not only if X and Y are surfaces, but if they are any simplicial complexes which may include, besides triangles, solid tetrahedra and higher dimensional 'simplexes'. This result is called the *topological invariance of homology groups*, and it has the following equivalent formulation: if, for some p, the pth homology group of X is not isomorphic to the pth homology group of Y, then X and Y are not homeomorphic. Thus homology groups can be used to distinguish objects from each other. It is not true in general that if the homology groups of X and Y are isomorphic then X and Y are homeomorphic: many different objects can have the same groups. This brings us back to the point made at the beginning of this Introduction, that many distinct geometrical situations are described by the same piece of algebra, and shows that homology groups are not the ultimate tool in distinguishing objects topologically from one another.

It happens that the applications in this book do not require the full invariance theorem quoted above, and accordingly this theorem is not proved here. Enough, in fact more than enough, is proved in Chapter 5 for our purposes. A sketch of the argument leading to the theorem of topological invariance is also included, together with references for the full proof.

The finite procedure for calculating homology groups can in practice be very lengthy, and for this reason general theorems are a help. A number of such theorems are given in Chapter 6; these enable the homology groups of all (closed) surfaces to be calculated without difficulty. Two more theorems, slightly more technical in nature and not strictly necessary for the applications, are given in Chapter 7.

Some of the work of Chapter 9 needs an 'unoriented' homology theory which differs in its details from the oriented homology theory studied in Chapters 1–7. This is presented in Chapter 8.

An Appendix contains all the group theory needed to read the book, presented in a fairly condensed form. Virtually no group theory is needed for Chapters 1–3.

Finally it should be emphasized that there are other approaches to homology theory. It is possible to define homology groups for far more general objects than simplicial complexes, in fact for arbitrary 'topological spaces' which may not be contained in a euclidean space at all. There are several methods for doing this, described as 'singular' or 'Čech' or 'Alexander–Spanier' homology. There is also a relatively recent development called 'Intersection Homology', designed to apply to spaces with singularities. These all have the property of topological invariance, so that they can sometimes be used to distinguish two spaces from one another – indeed this is one of the principal applications of homology theory. Again the method is not infallible: for example the three objects pictured below (cylinder, Möbius band, circle) all have the same groups in any homology theory, but no two are homeomorphic. Accounts of other theories can be found in, for example, the books of Spanier and of Eilenberg & Steenrod listed in the References. The latter classic text also presents a most attractive account of homology theory from the axiomatic point of view.

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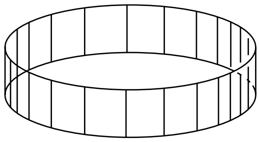
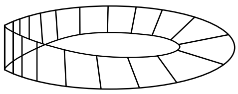
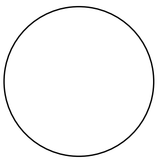


Cylinder



Möbius band



Circle

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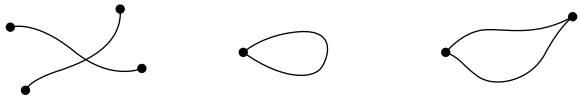
Graphs

The theory of graphs, otherwise known as networks, is a branch of mathematics which finds much application both within mathematics and in science. There are many books where the reader can find evidence to support this claim, for example those by Berge, Gross & Yellen, Diestel, Wilson & Beinecke and Bollobás listed in the References. Our motivation for touching on the subject here is different. Many of the ideas which we shall encounter later can be met, in a diluted form, in the simpler situation of graph theory, and that is the reason why we study graphs first. The problems studied by graph theorists are usually specifically applicable to graphs, and do not have sensible analogues in the more general theory of 'simplicial complexes' which we shall study. It is hardly surprising, therefore, that the theory we shall develop has little to say of interest to graph theorists, and it is partly for this reason that an occasional excursion is made, in this chapter and elsewhere, into genuine graph-theoretic territory. This may give the reader some idea of the difficult and interesting problems which lie there, but which the main concern of this book, namely homology theory, can scarcely touch.

Abstract graphs and realizations

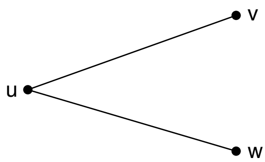
A graph is intuitively a finite set of points in space, called the vertices of the graph, some pairs of vertices being joined by arcs, called the edges of the graph. Two arcs are assumed to meet, if at all, in a vertex, and it is also assumed that no edge joins a vertex to itself and that two vertices are never connected by more than one edge. (Such a graph is often called a *simple* graph.) That is, we do not allow



The positions of the vertices and the lengths of the edges do not concern us; what is important is the number of vertices and the pairs of vertices which are connected by an edge. Thus the only information we need about an edge is the names of the vertices which it connects. This much information can be summed up in the following 'abstract' definition; we shall return to 'reality' shortly.

Definition 1.1. An *abstract graph* is a pair (V, E) where V is a finite set and E is a set of unordered pairs of distinct elements of V. Thus an element of E is of the form $\{v, w\}$ where v and w belong to V and $v \neq w$. The elements of V are called *vertices* and the element $\{v, w\}$ of E is called the *edge* joining v and w (or w and v).

Note that E may be empty: it is possible for no pair of vertices to be joined by an edge. Indeed V may be empty, in which case E certainly is – this is the empty graph with no vertices and no edges. In any case if V has n elements then E has at most $\binom{n}{2} = \frac{1}{2}n(n-1)$ elements. An abstract graph with the maximum number of edges is called *complete*: every pair of vertices is joined by an edge.



As an example of an abstract graph let $V = \{u, v, w\}$ and $E = \{\{u, v\}, \{u, w\}\}$. This abstract graph (V, E) can be pictured, or 'realized' as we shall say in a moment, by the diagram.

In many circumstances it is convenient to speak of graphs in which each edge has a direction or orientation (intuitively an arrow) attached to it. For example in an electrical circuit carrying direct current each edge (wire) has a direction, namely the direction in which current flows.

Definition 1.2. An *abstract oriented graph* is a pair (V, E) where V is a finite set and E is a set of ordered pairs of distinct elements of E with the property that, if $(v, w) \in E$ then $(w, v) \notin E$. The elements of V are called *vertices* and the element (v, w) of E is called the *edge* from v to w.

Associated with any abstract oriented graph there is an abstract graph, obtained by changing ordered to unordered pairs. The two graphs have the