# Random Graph Dynamics

The theory of random graphs began in the late 1950s in several papers by Erdös and Rényi. In the late twentieth century, the notion of six degrees of separation, meaning that any two people on the planet can be connected by a short chain of people who know each other, inspired Strogatz and Watts to define the small world random graph in which each site is connected to $k$ close neighbors, but also has long-range connections. At about the same time, it was observed in human social and sexual networks and on the Internet that the number of neighbors of an individual or computer has a power law distribution. This inspired Barabási and Albert to define the preferential attachment model, which has these properties. These two papers have led to an explosion of research. While this literature is extensive, many of the papers are based on simulations and nonrigorous arguments. The purpose of this book is to use a wide variety of mathematical argument to obtain insights into the properties of these graphs. A unique feature of this book is the interest in the dynamics of process taking place on the graph in addition to their geometric properties, such as connectedness and diameter.

Rick Durrett is a Professor of Mathematics at Cornell University. He received his Ph.D. in Operations Research from Stanford in 1976. After 9 years at UCLA, he moved to Cornell, where his research turned to applications of probability, first to ecology and, more recently, to genetics. He has written more than 150 papers, six other books, and has 33 academic descendants.

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# Preface

Chapter 1 will explain what this book is about. Here I will explain why I chose to write the book, how it is written, when and where the work was done, and who helped.

*Why.* It would make a good story if I was inspired to write this book by an image of Paul Erdös magically appearing on a cheese quesadilla, which I later sold for thousands of dollars on eBay. However, that is not true. The three main events that led to this book were (i) the use of random graphs in the solution of a problem that was part of Nathanael Berestycki's thesis; (ii) a talk that I heard Steve Strogatz give on the CHKNS model, which inspired me to prove some rigorous results about their model; and (iii) a book review I wrote on the books by Watts and Barabási for the *Notices of the American Math Society*.

The subject of this book was attractive for me, since many of the papers were outside the mathematics literature, so the rigorous proofs of the results were, in some cases, interesting mathematical problems. In addition, since I had worked for a number of years on the properties of stochastic spatial models on regular lattices, there was the natural question of how the behavior of these systems changed when one introduced long-range connections between individuals or considered power law degree distributions. Both of these modifications are reasonable if one considers the spread of influenza in a town where children bring the disease home from school, or the spread of sexually transmitted diseases through a population of individuals that have a widely varying number of contacts.

*How.* The aim of this book is to introduce the reader to the subject in the same way that a walk through Museé d'Orsay exposes the visitor to the many styles of impressionism. We will choose results to highlight the major themes, but we will not examine in detail every variation of preferential attachment that has been studied. We will concentrate on the ideas, giving the interesting parts of proofs, and referring the reader to the literature for the missing details. As Tom Liggett said after he had written his book *Interacting Particle Systems*, there is no point in having a book that is just a union of papers.

Throughout we approach the subject with a probabilistic viewpoint. One pragmatic reason is that, in the absence of futuristic procedures like the one Tom Cruise's character had in *The Minority Report*, these are the only eyes through which I can view the world. For connections to computer algorithms and their analysis, you will have to ask someone who knows that story. In addition, we will emphasize topics not found in other mathematical books. I have nothing to add to the treatment of random regular graphs in Janson, Luczak, and Rucińsky (2000), so I will not spend much time on this special case of random graphs with a fixed degree distribution. The classical theory of random graphs of Erdös and Rényi is covered nicely by Bollobás (2001), so I will keep treatment to the minimum necessary to prepare for more complicated examples.

Several reviewers lobbied for an introductory chapter devoted to some of the tools: branching processes, large deviations, martingales, convergence of Markov chains, almost exponentiality of waiting times, etc. Personally I do not think it is necessary (or even desirable) to read the entire *Kama Sutra* before having sex the first time, so instead I approach the book as I do my Ph.D. students' research projects. We will start looking at the subject and learn about the tools as they arise. Readers who find these interruptions distracting can note the statement and skip the proof, advice that can be applied to most of the results in the book.

*When and where.* The first version of these notes was written for a graduate seminar, which I gave on the topic at Cornell in the fall of 2004. On the Monday of the first full week of the semester, as I sat at my desk in Malott Hall, my back began to hurt, and I thought to myself that I had worked too hard on notes over the weekend. Two months, new desk chairs, a variety of drugs, physical therapy, and a lot of pain later, an MRI showed my problem was an infected disk. I still remember the radiologist's exciting words: "We can't let you go home until we get in touch with your doctor." Four days in the hospital and four months of sipping coffee every morning while an IV-dripped antibiotics into my arm, the bugs in me had been defeated, and I was back to almost normal. Reading papers on random graphs, figuring out proofs, and organizing the material while lying on my bed helped me to get through that ordeal.

In the summer of 2005, I revised the notes and added new material in preparation for six lectures on this topic, which I gave at the first Cornell Probability Summer School. Several more iterations of polishing followed. When my brain told me that the manuscript was in great shape, several paid reviewers showed me that there was still work to do. Finally, a month in Paris at École Normale Supérieure in February 2006 provided a pleasant setting for finishing the project. I would like to thank Jean Francois LeGall for the invitation to visit Paris, radio station 92.1 FM for providing the background music while I typed in my apartment, and the restaurants on and around Rue Mouffetard for giving me something to look forward to at the end of the day.

*Who.* If I were Sue Grafton, the title of this book would be "G is for Random Graphs." Continuing in the tradition of the first six books, I will update the story of my family by saying that my older son David is a freshman at Ithaca College studying journalism, while Greg is a senior who has been accepted at MIT and will go there to study computer science. With 8 years of tuition, room and board, and books to pay for in the next 5 years, I desperately need you to buy this book, or better yet put $50 in an envelope and mail it to me.

In the last two decades of diapers, ear infections, special education meetings, clarinet lessons, after-school activities, weekend music events, summer internships, and driving tests, my wife Susan Myron has been the one with the more difficult job. There are no words that can adequately convey my happiness after 25 years of marriage, except that I am hoping for many more.

I would like to thank Mark Newman for his good-natured answers to several random e-mails. Postdocs Paul Jung and Lea Popovic read several of the early chapters in detail and made a number of useful suggestions. The anonymous reviewers who each read one or two chapters helped illuminate the dark corners of the manuscript and contributed some useful insights. Lauren Cowles did a wonderful job of managing the process, and the book is much better for her efforts.

As usual, I look forward to your constructive criticisms and corrections by e-mail to rtd1@cornell.edu and you can look for lists of typos, etc., on my Web page:

www.math.cornell.edu/~durrett

There you can find also copies of my recent papers, most of which concern probability problems that arise from biology.