1
p-adic numbers

1.1. Introduction

In this chapter some basic facts on the field of p-adic numbers $\mathbb{Q}_p$ are presented. Here we follow some sections from the books [47], [96], [98], [152], [241], and especially from the textbook [112]. Section 1.9.3 follows [254] and [241, 1.6.]. Section 1.9.4 is based on [163] and [162, 2.4.].

1.2. Archimedean and non-Archimedean normed fields

Denote by $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$ the sets of positive integers, rational numbers, integers, real numbers, nonnegative real numbers and complex numbers, respectively, and set $\mathbb{N}_0 = 0 \cup \mathbb{N}$.

Definition 1.1. Let X be a nonempty set. A *distance* or *metric* on X is a map $d : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$ we have

(1) $d(x, y) = 0 \Leftrightarrow x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \le d(x, z) + d(z, y)$ (*triangle inequality*).

A set together with a metric is called a *metric space*.

A metric d is called *non-Archimedean* (or *ultra-metric*) if it satisfies the additional condition

(3′) $d(x, y) \le \max(d(x, z), d(z, y))$ (*strong triangle inequality*).

The corresponding metric space is called an *ultrametric space*.

Since $\max(d(x, z), d(z, y))$ does not exceed the sum $d(x, z) + d(z, y)$, condition (3′), the *strong triangle inequality*, implies condition (3), the *triangle inequality*.

The same set X can give rise to many different metric spaces (X, d).

Definition 1.2. (i) Let X be a metric space with respect to metric d. A sequence $\{x_n : x_n \in X\}$ is called a *Cauchy sequence* if for any $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n > N(\varepsilon)$, i.e.,

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0.$$

(ii) If any Cauchy sequence $\{x_n\}$ in the metric space X has a limit in X, then X is called a *complete metric space*.

(iii) A subset $M \subset X$ is called *dense* in X if every open ball $U_r^-(a) = \{x \in X : d(x, a) < r\}$ around every element $a \in X$ contains an element of M, i.e., for every $a \in X$ and every $r > 0$ we have $U_r^-(a) \cap M \neq \emptyset$.

If X is not a complete metric space, it can be completed by using an explicit construction of the completion. The proof of this important theorem can be found in many textbooks on functional analysis. This proof consists of an *explicit construction of the completion* $\widehat{X}$ and the metric $\widehat{d}$ on it.

In Section 1.4, we will give the complete proof of this theorem in the particular case of metric spaces called *normed fields*.

Definition 1.3. We say that two metrics d_1 and d_2 on a metric space X are *equivalent* if a Cauchy sequence with respect to d_1 is also a Cauchy sequence with respect to d_2, and vice versa.

The sets X we will be dealing with will mostly be fields. Recall that a field F is a set together with two operations $+$ and $\cdot$ such that F is a commutative group under $+$ and $F^\times = F \setminus \{0\}$ is a commutative group under $\cdot$, and the distributive law holds. $F^\times$ is called the *multiplicative group of the field*.

Denote by $\mathbb{Z}(F)$ the ring generated in F by its unity element. If F has zero characteristic, Char $F = 0$, i.e., for any $n = 1, 2, \dots$

$$n \cdot 1 = \underbrace{1 + \cdots + 1}_{n \text{ times}} \in F, \quad n \cdot 1 \neq 0,$$

then $\mathbb{Z}(F)$ is isomorphic to the ring of integers $\mathbb{Z}$. Therefore in this case we can consider $\mathbb{Z}$ as a subring of the field F. We shall denote the element $n \cdot 1$ by the same symbol n as the corresponding natural number. In what follows we consider only normed rings F which have *zero characteristic*.

Definition 1.4. Let F be a field. A *norm* on F is a map $\|\cdot\| : F \to \mathbb{R}_+$ such that for all $x, y \in F$ we have

(1) $\|x\| = 0 \Leftrightarrow x = 0$;

Cambridge University Press
978-0-521-14856-6 - Theory of p-adic Distributions: Linear and Nonlinear Models
S. Albeverio, A. Yu. Khrennikov and V. M. Shelkovich
Excerpt
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(2) $\|xy\| = \|x\|\,\|y\|$[1];
(3) $\|x + y\| \le \|x\| + \|y\|$ (*triangle inequality*).
A norm $\|\cdot\|$ is called *non-Archimedean* if it satisfies the additional condition
(3′) $\|x + y\| \le \max(\|x\|, \|y\|)$ (*strong triangle inequality*).

The norm $\|\cdot\|$ is called *trivial* if $\|0\| = 0$ and $\|x\| = 1$ for all $x \ne 0$.

Using a norm $\|\cdot\|$, one can introduce a metric $d(x, y) = \|x - y\|$ which is induced by this norm. In this case we can regard the field F as a metric space. It is easy to see that a metric induced by a non-Archimedean norm is also non-Archimedean.

Definition 1.5. We say that two norms $\|x\|_1$ and $\|x\|_2$ on a normed field F are *equivalent* ($\|x\|_1 \sim \|x\|_2$) if they induce equivalent metrics.

Theorem 1.2.1. ([47, Ch. I,3.], [98, 3.1.]) *Let* $\|x\|_1 \sim \|x\|_2$.
(i) *If* $\|x\|_1$ *is trivial then* $\|x\|_2$ *is also trivial.*
(ii) $\|x\|_1 < 1$ *if and only if* $\|x\|_2 < 1$; $\|x\|_1 > 1$ *if and only if* $\|x\|_2 > 1$; $\|x\|_1 = 1$ *if and only if* $\|x\|_2 = 1$.

Theorem 1.2.2. *Let* $\|x\|_1$ *and* $\|x\|_2$ *be two norms on a field* F. *Then* $\|x\|_1 \sim \|x\|_2$ *if and only if there exists a positive real* α *such that*

$$\|x\|_2 = \|x\|_1^{\alpha}, \quad \forall x \in F. \tag{1.2.1}$$

Proof. Let $\|x\|_1 \sim \|x\|_2$. If $\|x\|_1$ is trivial, then, according to Theorem 1.2.1, $\|x\|_2$ is also trivial. Consequently, (1.2.1) is satisfied for any α.

Suppose that $\|x\|_1$ is non-trivial. In this case there exists an element $a \in F$ such that $\|a\|_1 \ne 1$. If necessary, one can replace a by a^{-1}, and consequently, one can assume that $\|a\|_1 < 1$. Let us define

$$\alpha = \frac{\log \|a\|_2}{\log \|a\|_1}.$$

According to Theorem 1.2.1, $\|a\|_1 < 1$ implies $\|a\|_2 < 1$. Thus $\alpha > 0$.

Now we will show that α satisfies (1.2.1). Choosing $x \in F$ such that $\|x\|_1 < 1$, we consider the set

$$S = \{r = \frac{m}{n} : m, n \in \mathbb{N},\ \|x\|_1^r < \|a\|_1\}. \tag{1.2.2}$$

For any $r \in S$ we have $\|x\|_1^m < \|a\|_1^n$, i.e., $\|\frac{x^m}{a}x^n\|_1 < 1$. In view of Theorem 1.2.1, we conclude that $\|\frac{x^m}{a}x^n\|_2 < 1$, i.e., $\|x\|_2^m < \|a\|_2^n$ and $\|x\|_2^r < \|a\|_2$.

[1] In general, instead of this axiom the following one is used: $\|xy\| \le \|x\|\,\|y\|$. But in this book to define a norm on the field we will use axiom (2).

Using the same arguments, one can see that

$$S = \{r = \frac{m}{n} : m, n \in \mathbb{N},\ \|x\|_2^r < \|a\|_2\}. \tag{1.2.3}$$

The conditions (1.2.2) and (1.2.3) can be rewritten as

$$r > \frac{\log \|a\|_1}{\log \|x\|_1}, \qquad r > \frac{\log \|a\|_2}{\log \|x\|_2}. \tag{1.2.4}$$

The inequalities (1.2.4) imply that

$$\frac{\log \|a\|_1}{\log \|x\|_1} = \frac{\log \|a\|_2}{\log \|x\|_2}.$$

Otherwise there would be a rational r between these two numbers and only one of the conditions in (1.2.4) would be satisfied. Consequently,

$$\frac{\log \|x\|_2}{\log \|x\|_1} = \frac{\log \|a\|_2}{\log \|a\|_1} = \alpha,$$

i.e., (1.2.1) holds.

The cases $\|x\|_1 > 1$ and $\|x\|_1 = 1$ follow from Theorem 1.2.1. □

Proposition 1.2.3. *The norm* $\|x\| = |x|^\alpha$, $\alpha > 0$, *is a norm on* $\mathbb{Q}$ *if and only if* $\alpha \le 1$. *In this case it is equivalent to the norm* $|x|$.

Proof. Let $\alpha \le 1$. It is clear that the first two properties of the norm from Definition 1.4 hold. Let us examine the third property (the triangle inequality). Suppose that $|y| \le |x|$. Then

$$|x + y|^\alpha \le \big(|x| + |y|\big)^\alpha = |x|^\alpha \Big(1 + \frac{|y|}{|x|}\Big)^\alpha$$

$$\le |x|^\alpha \Big(1 + \frac{|y|}{|x|}\Big) \le |x|^\alpha \Big(1 + \frac{|y|^\alpha}{|x|^\alpha}\Big) \le |x|^\alpha + |y|^\alpha.$$

If $\alpha > 1$, the triangle inequality does not hold. For example, $|1 + 1|^\alpha = 2^\alpha > |1|^\alpha + |1|^\alpha = 2$. □

Now we give a criterion for a norm to be non-Archimedean.

Theorem 1.2.4. *The norm* $\|\cdot\|$ *is non-Archimedean if and only if* $\|n\| \le 1$ *for any* $n \in \mathbb{Z}$.

Proof. We prove this proposition by induction. Let the norm $\|\cdot\|$ be non-Archimedean. It is clear that $\|1\| = 1 \le 1$. Let us assume that $\|k\| = 1 \le 1$ for all $k = 1, 2, \dots, k - 1$. Next, it follows from our assumption that $\|n\| = \|1 + (n - 1)\| \le \max(\|1\|, \|(n - 1)\|) = 1$. Thus, according to the induction axiom, we have $\|n\| \le 1$ for any $n \in \mathbb{N}$. Since $\|-n\| = \|n\|$, we conclude that $\|n\| \le 1$ for any $n \in \mathbb{Z}$.

Conversely, assume that $\|n\| \le 1$ for any $n \in \mathbb{Z}$. Since the binomial coefficients $C_n^k = \frac{n!}{k!(n-k)!}$, $k \le n$, are integers, we have $\|C_n^k\| \le 1$. Thus

$$\|x+y\|^n = \|(x+y)^n\| = \Big\|\sum_{k=0}^{n} C_n^k x^k y^{n-k}\Big\| \le \sum_{k=0}^{n} \big\|C_n^k\big\| \|x\|^k \|y\|^{n-k}$$
$$\le \sum_{k=0}^{n} \|x\|^k \|y\|^{n-k} \le (n+1)\big(\max(\|x\|, \|y\|)\big)^n.$$

Hence, for every integer n we have $\|x+y\| \le \sqrt[n]{n+1}\max(\|x\|, \|y\|)$. Letting n go to ∞, we obtain $\|x+y\| \le \max(\|x\|, \|y\|)$, i.e., the norm $\|\cdot\|$ is non-Archimedean. □

Corollary 1.2.5. *The norm $\|\cdot\|$ is non-Archimedean if and only if* $\sup\{\|n\| : n \in \mathbb{Z}\} = 1$.

By Theorem 1.2.4 one can observe the difference between non-Archimedean and Archimedean norms. According to this theorem, a norm $\|\cdot\|$ is Archimedean if and only if it satisfies the *Archimedean property* (*axiom*): *for any $x, y \in F$, $x \ne 0$, there exists $n \in \mathbb{N}$ such that $\|nx\| \ge \|y\|$.*

Indeed, if $\|y\| > \|x\|$, then the Archimedean property implies the existence of $n \in \mathbb{N}$ such that $\|n\| > \|y\|/\|x\| > 1$, i.e., the norm is Archimedean. Conversely, if a norm $\|\cdot\|$ is Archimedean, there exists $n \in \mathbb{N}$ such that $\|n\| > 1$. Hence, $\|n\|^k \to \infty$, as $k \to \infty$. Consequently, for any $x, y \in F$, $x \ne 0$, there exists k such that $\|n^k\| > \|y\|/\|x\|$. Thus the Archimedean property $\|n^k x\| > \|y\|$ holds.

If the norm is non-Archimedean, for any $n \in \mathbb{Z}$ we have $\|nx\| \le \|x\|$.

The non-Archimedean property of a norm has some strange consequences.

Proposition 1.2.6. *If a field F is non-Archimedean, then for $x, y \in F$*

$$\|x\| \ne \|y\| \Longrightarrow \|x+y\| = \max\big(\|x\|, \|y\|\big).$$

Thus, any triangle in an ultra-metric space is isosceles and the length of its base does not exceed the lengths of the sides.

Proof. Exchanging x and y if necessary, we may suppose that $\|x\| > \|y\|$. By the strong triangle inequality (3′) in Definition 1.4,

$$\|x+y\| \le \max(\|x\|, \|y\|) = \|x\|.$$

On the other hand,

$$\|x\| = \|(x+y) - y\| \le \max(\|x+y\|, \|y\|).$$

Since $\|x\| > \|y\|$, we must have $\|x\| \le \|x+y\|$. Consequently, $\|x\| = \|x+y\|$. □

Thus for a non-Archimedean field $\|x \pm y\| \le \max(\|x\|, \|y\|)$, and in the case where $\|x\| \ne \|y\|$, the above inequality becomes an equality.

Proposition 1.2.7. *If the field F is non-Archimedean, then any point of an open ball $U_r^-(a) = \{x \in F : \|x-a\| < r\}$ is its center, i.e., if $b \in U_r^-(a)$, then $U_r^-(b) = U_r^-(a)$. The same statement is true for a closed ball $U_r(a) = \{x \in F : \|x-a\| \le r\}$.*

Proof.

$$x \in U_r^-(a) \Rightarrow \|x-a\| < r \Rightarrow \|x-b\| = \|(x-a)+(a-b)\| \\ \le \max(\|x-a\|, \|a-b\|) < r \Rightarrow x \in U_r^-(b).$$

The reverse implication is proved in the same way.

Replacing $<$ with $\le$, we prove the statement for a closed ball $U_r(a)$. □

1.3. Metrics and norms on the field of rational numbers

1.3.1. p-adic norm

We know that there exists a norm of the field $\mathbb{Q}$: the ordinary absolute value $|\cdot|$. This norm induces the ordinary Euclidean metric $d(x, y) = |x-y|$. The question arises: are there any other norms on $\mathbb{Q}$? It turns out that *there are other norms*. Below we describe all such norms.

Definition 1.6. Let p be a prime number. We define the *p-adic order* $\mathrm{ord}_p(x)$ *of a rational number* $x \in \mathbb{Q}$ by the following definition:

(i) If $x \in \mathbb{Z}$, then $\mathrm{ord}_p(x)$ is equal to the highest power of p which divides x.

(ii) If $x = a/b$, where $a, b \in \mathbb{Z}$, then $\mathrm{ord}_p(x) = \mathrm{ord}_p(a) - \mathrm{ord}_p(b)$. The p-adic order of $x \in \mathbb{Q}$ is also called the *p-adic additive valuation* and denoted as $v_p(x)$.

(iii) We set $\mathrm{ord}_p(0) = +\infty$.

The reason to set $\mathrm{ord}_p(0) = +\infty$ is that we can divide 0 by p^n for each $n \in \mathbb{N}$.

It is clear that for all $x, y \in \mathbb{Q}$:

$$\begin{aligned} \mathrm{ord}_p(xy) &= \mathrm{ord}_p(x) + \mathrm{ord}_p(y), \\ \mathrm{ord}_p(x+y) &\ge \min\big(\mathrm{ord}_p(x), \mathrm{ord}_p(y)\big). \end{aligned} \tag{1.3.1}$$

Now we define a map $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_+$ as follows:

$$|x|_p = \begin{cases} p^{-\mathrm{ord}_p(x)}, & \text{if} \quad x \ne 0, \\ 0, & \text{if} \quad x = 0. \end{cases} \tag{1.3.2}$$

We point out that the definition $|0|_p = 0$ follows from Definition 1.6 (iii).

It is clear that the function $|\cdot|_p$ can take only a discrete set of values $\{p^\gamma : \gamma \in \mathbb{Z}\}$.

Note that if $x, y \in \mathbb{N}$, then $x \equiv b\,(\mathrm{mod}\, p^n)$ if and only if $|x-y|_p \le p^{-n}$. By the notation $x \equiv b\,(\mathrm{mod}\, z)$ we mean that z divides $x - y$.

Theorem 1.3.1. *The map $|\cdot|_p$ is an non-Archimedean norm on the field of rational numbers $\mathbb{Q}$, i.e., it satisfies the axioms* (1), (2), (3′) *from Definition* 1.4.

Proof. It is clear that axiom (1) holds. Because $\mathrm{ord}_p(xy) = \mathrm{ord}_p x + \mathrm{ord}_p y$, axiom (2) also holds.

Let us verify axiom (3′). If $x = 0$ or $y = 0$, or $x + y = 0$, property (3′) is trivial, so we assume that $x, y, x+y$ are nonzero. Let $x = a/b$, $y = c/d$ be written in lowest terms. Then we have $x + y = (ad + bc)/bd$ and

$$\begin{aligned} \mathrm{ord}_p(x+y) &= \mathrm{ord}_p(ad+bc) + \mathrm{ord}_p(bd) \\ &\ge \min\big(\mathrm{ord}_p(ad), \mathrm{ord}_p(bc)\big) - \mathrm{ord}_p(b) - \mathrm{ord}_p(d) \\ &= \min\big(\mathrm{ord}_p(a) + \mathrm{ord}_p(d), \mathrm{ord}_p(b) + \mathrm{ord}_p(c)\big) - \mathrm{ord}_p(b) \\ &\quad -\mathrm{ord}_p(d) \\ &= \min\big(\mathrm{ord}_p(a) - \mathrm{ord}_p(b), \mathrm{ord}_p(c) - \mathrm{ord}_p(d)\big) \\ &= \min\big(\mathrm{ord}_p(x), \mathrm{ord}_p(y)\big). \end{aligned}$$

Consequently,

$$\begin{aligned} |x+y|_p = p^{-\mathrm{ord}_p(x+y)} &\le \max\big(p^{-\mathrm{ord}_p(x)}, p^{-\mathrm{ord}_p(y)}\big) \\ &= \max\big(|x|_p, |y|_p\big) \le |x|_p + |y|_p. \end{aligned}$$

□

Remark 1.1. If in definition (1.3.2) instead of a prime number p we use an arbitrary integer number $m > 1$, then axiom (2) from Definition 1.4 may fail. For example, let $m = 4$. Then $|2|_4 = 1$, but $|2 \cdot 2|_4 = \frac{1}{4} \ne |2|_4\,|2|_4 = 1$.

Remark 1.2. According to Proposition 1.2.6, if $|x|_p \neq |y|_p$ then

$$|x+y|_p = \max\big(|x|_p, |y|_p\big).$$

For $p=2$, if $|x|_2 = |y|_2$ then

$$|x+y|_2 \le \frac{1}{2}|x|_p.$$

The latter inequality follows from definition (1.3.2).

It is clear that $|p^n|_p \to 0$ as $n \to \infty$, so that *high* powers of p are *small* with respect to the p-adic norm (1.3.2).

In view of Theorem 1.2.4, for any $n \in \mathbb{Z}$ we have $|nx|_p \le |x|_p$.

Remark 1.3. The norms $|\cdot|_{p_1}$ and $|\cdot|_{p_2}$ are not equivalent if p_1 and p_2 are different primes. Indeed, for the sequence $x_n = (p_1/p_2)^n$ we have $|x_n|_{p_1} \to 0$, but $|x_n|_{p_1} \to \infty$, as $n \to \infty$.

Thus we observe that the field $\mathbb{Q}$ admits the p-adic norms $|\cdot|_p$ for each prime p, as well the ordinary norm (ordinary absolute value) $|\cdot|$. The last norm sometimes is denoted by $|\cdot|_\infty$ for $p=\infty$, and $p=\infty$ sometimes is called the infinite prime.

1.3.2. The Ostrovski theorem

Now we prove that *there are no other norms on* $\mathbb{Q}$ except for $|\cdot|_p$ and $|\cdot|$.

Theorem 1.3.2. (*Ostrovski's theorem*) *Any* non-trivial norm $\|\cdot\|$ *on the field* $\mathbb{Q}$ *is equivalent either to the real norm* $|\cdot|$ *or to one of the* p*-adic norms* $|\cdot|_p$.

Proof. 1. Suppose that $\|\cdot\|$ is Archimedean, i.e., there exists $n \in \mathbb{N}$ such that $\|n\| > 1$. Let n_0 be the least such n. Of course we can find some positive real number α so that $\|n_0\| = n_0^\alpha$.

Let us write any positive integer n in base n_0, i.e., in the form

$$n = a_0 + a_1 n_0 + \cdots + a_s n_0^s, \qquad (1.3.3)$$

where $0 \le a_i \le n_0 - 1$, $i = 0, 1, \dots, s$, $a_s \neq 0$. Then

$$\|n\| \le \|a_0\| + \|a_1 n_0\| + \cdots + \|a_s n_0^s\| = \|a_0\| + \|a_1\| n_0^\alpha + \cdots + \|a_s\| n_0^{s\alpha}.$$

Since we chose n_0 to be the smallest integer whose norm was greater than 1, we know that $\|a_i\| \le 1$. Hence

$$\|n\| \le 1 + n_0^\alpha + \cdots + n_0^{s\alpha} = n_0^{s\alpha}\big(1 + n_0^{-\alpha} + \cdots + n_0^{-s\alpha}\big)$$
$$\le n_0^{s\alpha} \sum_{i=0}^{\infty} n_0^{-i\alpha} = n_0^{s\alpha} \frac{n_0^\alpha}{n_0^\alpha - 1}.$$

Since $n \geq n_0^s$, the latter inequality implies

$$\|n\| \leq C n_0^{s\alpha} \leq C n^\alpha,$$

where $C = \frac{n_0^\alpha}{n_0^\alpha - 1}$ does not depend on n. Substituting an integer of the form n^N in the above inequality instead of n, we have

$$\|n^N\| \leq C n^{N\alpha} \Leftrightarrow \|n\| \leq \sqrt[N]{C} n^\alpha.$$

Letting $N \to \infty$ for n fixed, we obtain

$$\|n\| \leq n^\alpha. \tag{1.3.4}$$

Now we prove the opposite inequality. It follows from (1.3.3) that $n_0^s \leq n < n_0^{s+1}$, and, consequently,

$$n_0^{(s+1)\alpha} = \|n_0^{s+1}\| = \|n + n_0^{s+1} - n\| \leq \|n\| + \|n_0^{s+1} - n\|.$$

Thus

$$\|n\| \geq \|n_0^{s+1}\| - \|n_0^{s+1} - n\| \geq n_0^{(s+1)\alpha} - (n_0^{s+1} - n)^\alpha,$$

because according to (1.3.4), we have $\|n_0^{s+1} - n\| \leq (n_0^{s+1} - n)^\alpha$. Now since $n_0^s \leq n < n_0^{s+1}$, it follows that

$$\begin{aligned} \|n\| \geq n_0^{(s+1)\alpha} - (n_0^{s+1} - n)^\alpha &= n_0^{(s+1)\alpha}\left(1 - \left(1 - \frac{1}{n_0}\right)^\alpha\right) \\ &= C' n_0^{(s+1)\alpha} \geq C' n_0^\alpha, \end{aligned}$$

and once again $C' = 1 - \left(1 - \frac{1}{n_0}\right)^\alpha$ does not depend on n and is positive. As before, we now use the latter inequality for n^N, take the Nth root, and let $N \to \infty$, obtaining

$$\|n\| \geq n^\alpha. \tag{1.3.5}$$

From (1.3.4) and (1.3.5), we deduce that $\|n\| = n^\alpha$ for all $n \in \mathbb{N}$. Using property (2) of the norm, one can see that $\|x\| = |x|^\alpha$ for all $x \in \mathbb{Q}$. In view of Theorem 1.2.2, we can conclude that such a norm is equivalent to the ordinary absolute value $|\cdot| \equiv |\cdot|_\infty$.

2. Now suppose that $\|\cdot\|$ is non-Archimedean. Then according to Theorem 1.2.4, we have $\|n\| \leq 1$ for every $n \in \mathbb{N}$. Since $\|\cdot\|$ is non-trivial, there exists a smallest $n_0 \in \mathbb{N}$ such that $\|n_0\| < 1$. First, we observe that n_0 must be a prime number. Indeed, if $n_0 = n_1 n_2$, where n_1 and n_2 are smaller than n_0, then by our choice for n_0 we would have $\|n_1\| = \|n_2\| = 1$, and so $\|n_0\| = \|n_1\|\|n_2\| = 1$. Thus n_0 is a prime number. Denote this prime number by p.

Next, we will prove that if $n \in \mathbb{Z}$ is not divisible by p, then $\|n\| = 1$. If we divide n by p we will have a remainder, so that we can write $n = rp + s$, where $0 < s < p$. By the minimality of p, $\|s\| = 1$. We also have $\|rp\| < 1$, because $\|p\| < 1$ and $\|r\| \le 1$ (since r is an integer). Consequently,

$$\|n - s\| = \|rp\| < \|s\| = 1,$$

and by Proposition 1.2.6, $\|n\| = \|s\| = 1$. Finally, given $n \in \mathbb{Z}$, one can write $n = p^v n'$, where p does not divide n'. Hence

$$\|n\| = \|p^v n'\| = \|p\|^v,$$

where $\rho = \|p\| < 1$. Then $\rho = (1/p)^\alpha$ for some positive real α. Therefore

$$\|n\| = (1/p)^{v\alpha} = \|n\|_p^\alpha.$$

Now, using property (2) of the norm, it is easy to see that the same formula holds for any nonzero rational number x in place of n. In view of Theorem 1.2.2, we have $|\cdot| \equiv |\cdot|_p$.

Thus the theorem is proved. □

Proposition 1.3.3. *For any $x \in \mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$ the following relation holds:*

$$\prod_{2 \le p \le \infty} |x|_p = 1$$

Proof. Expanding a rational number x by prime factors $x = \epsilon p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where $\epsilon = \pm 1$, p_j are different prime numbers, $\alpha_j \in \mathbb{Z}$, and using definition (1.3.2) of p-adic norm and the ordinary absolute value, we obtain

$$|x|_{p_j} = p_j^{-\alpha_j}; \qquad |x|_p = 1, \;\; p \ne p_j; \qquad |x|_\infty = p_1^{-\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}.$$

These facts imply our statement. □

The formula from Proposition 1.3.3 establishes the relation between all non-trivial norms of $\mathbb{Q}$ [2].

1.4. Construction of the completion of a normed field

Starting from an arbitrary normed field $(F, \|\cdot\|)$ not necessary complete with respect to its norm $\|\cdot\|$, we construct the field $\widehat{F}$ containing F, and supply it with a norm (induced by the norm $\|\cdot\|$ of F) in such a way that the field

[2] Formulas of this type are called *"adelic products"*. For details on adelic formulas, see for example [233]–[235], [244].