1

Introduction

1.1 Phase space, phase portrait

In this first chapter we give an introduction to the stability of discrete systems and bifurcations from the geometrical viewpoint of the theory of dynamical systems in phase space. In the first part, which is more mathematical than physical, we define the fundamental ideas. These ideas are then illustrated by examples borrowed from hydrodynamics and the physics of liquids. We close the chapter with a brief presentation of the idea of transient growth, which is related to nonorthogonality of the eigenvectors of a linear system.

The time evolution of a discrete (noncontinuous) physical system is generally governed by differential equations following from physical conservation principles and the laws describing the phenomenological behavior. These equations can often be written as a system of first-order *ordinary differential equations* (ODEs) of the form (see, e.g., Glendinning (1994)):

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = X_i(x_1, \ldots, x_n, t), \qquad i = 1, \ldots, n. \tag{1.1}$$

The remainder of this chapter will consider only *autonomous* systems, in which time does not appear explicitly on the right-hand side. The variables x_i are called the *degrees of freedom of the system.*[1] As an example, let us consider a simple damped nonlinear pendulum whose vertical position is specified by the angle θ. Its equation of motion

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \mu\frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega_0^2 \sin\theta = 0 \tag{1.2}$$

[1] The degrees of freedom in question are the dynamical degrees of freedom (here, the position and velocity), which are different from the kinematical degrees of freedom in physical space (the positions).

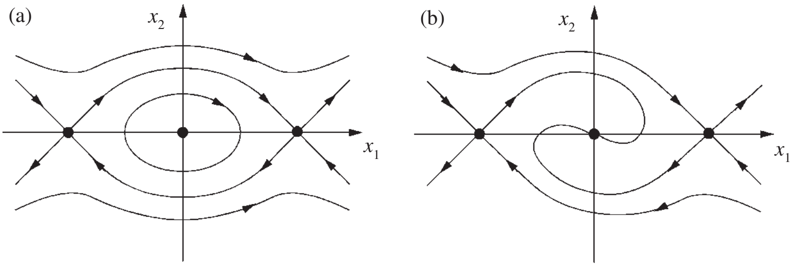

Figure 1.1 Phase portraits of the oscillator (1.3) for (a) $\mu = 0$; (b) $\mu > 0$.

can be written equivalently as a system of two ODEs by setting $x_1 = \theta$, $x_2 = \mathrm{d}\theta/\mathrm{d}t$:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = -\mu x_2 - \omega_0^2 \sin x_1. \tag{1.3}$$

Any solution of a system of ODEs for a given initial condition can be represented by a curve in the space of the degrees of freedom, called the *phase space*. For the system (1.3) the phase space is the (x_1, x_2) plane. Figure 1.1 shows typical trajectories corresponding to given initial conditions for $\mu = 0$ and $\mu > 0$. The case $\mu = 0$ corresponds to a nondissipative oscillator (i.e., where the mechanical energy remains constant), and the case $\mu > 0$ corresponds to a dissipative oscillator (where the mechanical energy decreases over time). A representation of this type which depicts the essential features of the solutions of a system of ODEs is called the *phase portrait*, which allows the trajectory to be plotted qualitatively for any given initial condition. We use the term *dynamical system* to refer to any system of ODEs studied from the viewpoint of obtaining the phase portrait of the system.

The phase portrait can be guessed easily for a system as elementary as the pendulum (1.3). For more complicated systems the first step is to determine the fixed points and study their stability. When there are several fixed points the second important step is to determine to which fixed point the system evolves for various initial conditions. The ensemble of initial conditions resulting in motion to a particular fixed point is called the *basin of attraction* of that fixed point.

1.2 Stability of a fixed point

1.2.1 Fixed points

The *equilibrium states* of a physical system correspond to the stationary solutions of the system of ODEs, defined as

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = 0, \qquad i = 1, \ldots, n.$$

These solutions are represented in phase space by points called *fixed points*. The fixed points are determined by solving the nonlinear system

$$X_i(x_1,\dots,x_n)=0, \qquad i=1,\dots,n.$$

The fixed points of the system (1.3) are $(x_1,x_2)=(0,0)$ and $(x_1,x_2)=(\pi,0)$ (modulo 2π). In the case of a system where the forces acting can be derived from a potential $V(x_1,\dots,x_n)$, or are proportional to velocities (viscous or friction forces), the equilibrium states correspond to the extrema of the potential (Landau and Lifshitz, 1976).

1.2.2 Linear stability of a fixed point

Once the fixed points are determined, the question of their stability (i.e., the stability of the corresponding equilibrium states) arises. When these equilibrium states are the extrema of a potential, the states of stable and unstable equilibrium correspond respectively to the minima and maxima of the potential (Landau and Lifshitz, 1976), and knowledge of the potential is sufficient for sketching the phase portrait. For example, the phase portrait of the system (1.3) for $\mu=0$ can easily be drawn by noticing that the only force involved in the equation of motion, the weight, can be derived from the potential $V(\theta)=-mg\cos\theta$. When there is no such potential, a general method based on linear algebra can be used to study the stability of a fixed point with respect to small perturbations. Accordingly, let us consider the system (1.1) written in vector form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}=\mathbf{X}(\mathbf{x}), \qquad \text{where} \qquad \mathbf{x}=(x_1,\dots,x_n),$$

which has a fixed point at $\mathbf{x}=\mathbf{a}$. The idea is that for small perturbations from equilibrium of amplitude $\epsilon \ll 1$, the smooth function $\mathbf{X}$ can be expanded about the fixed point in a Taylor series, and all products of perturbations can be neglected because they are of order ϵ^2 or smaller. Setting $\mathbf{y}=\mathbf{x}-\mathbf{a}$, the resulting linearized system is written as

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}=\mathbf{L}(\mathbf{a})\mathbf{y}, \tag{1.4}$$

where $\mathbf{L}(\mathbf{a})$ is the Jacobian matrix of $\mathbf{X}(\mathbf{x})$ calculated at the point $\mathbf{a}$, the elements of which are $L_{ij}=\partial X_i/\partial x_j(\mathbf{a})$. When, as in the present case of autonomous systems, the elements L_{ij} are independent of time, the system (1.4) is linear with constant coefficients and its solutions are exponentials $\exp(st)$. The problem then becomes an algebraic eigenvalue problem $\mathbf{L}(\mathbf{a})\mathbf{y}=s\mathbf{y}$, which has a nontrivial solution only if the determinant of $\mathbf{L}-s\mathbf{I}$ vanishes, where $\mathbf{I}$ is the unit matrix. This determinant is a polynomial in s, called the *characteristic polynomial*, and its roots are the *eigenvalues*. If the real parts of the eigenvalues are all negative, the solution is a sum of decaying exponentials, and any perturbation from equilibrium dies out at large times: the fixed point is asymptotically stable. However, if at least one of the eigenvalues has positive real part, the fixed point is unstable. To study the *linear stability* of a fixed point we therefore need to (i) find the eigenvalues of the linearized problem, (ii) find the eigenvectors or eigendirections in the phase space, and (iii) plot the phase portrait in the neighborhood of the fixed point.

In two dimensions the classification of types of fixed point is simple. The characteristic polynomial $\det(\mathbf{L}-sI)$ depends only on the trace $\mathrm{tr}(\mathbf{L})$ and the determinant $\det(\mathbf{L})$ of the matrix $\mathbf{L}$:

$$\det(\mathbf{L}-sI)=s^2-\mathrm{tr}(\mathbf{L})\,s+\det(\mathbf{L}). \tag{1.5}$$

The various cases, illustrated in Figure 1.2, are the following:

- $\det(\mathbf{L})<0$: s_1 and s_2 are real and have opposite signs; the trajectories are hyperbolas whose asymptotes are the eigendirections, and the fixed point is called a *saddle* (Figure 1.2a).
- $\det(\mathbf{L})>0$ and $4\det(\mathbf{L})\le \mathrm{tr}^2(\mathbf{L})$ (positive or zero discriminant): s_1 and s_2 are real and have the same sign as $\mathrm{tr}(\mathbf{L})$; the fixed point is called a *node*, and is attractive (stable) if $\mathrm{tr}(\mathbf{L})<0$ or repulsive (unstable) if $\mathrm{tr}(\mathbf{L})>0$ (Figure 1.2b). If the discriminant is zero, s is a double root and two cases can be distinguished: either $\mathbf{L}$ is a multiple of the identity $\mathbf{I}$, in which case the trajectories are straight lines and the node is called a *star*, or $\mathbf{L}$ is nondiagonalizable and the node is

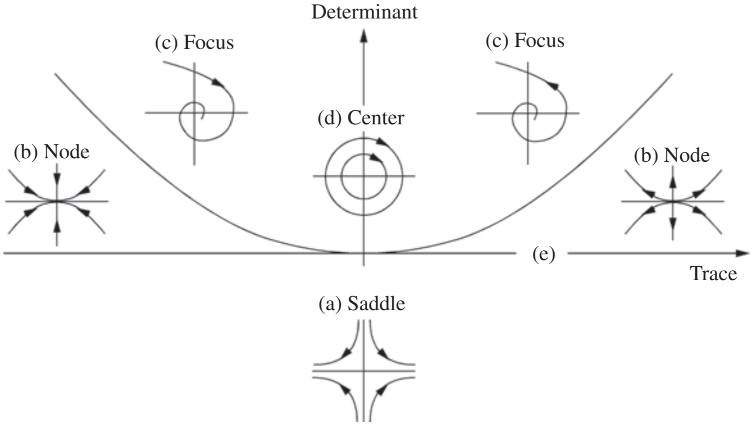

Figure 1.2 Types of fixed point in $\mathbf{R}^2$. The parabola corresponds to $\mathrm{tr}^2\mathbf{L}-4\det\mathbf{L}=0$ (discriminant of the characteristic polynomial equal to zero).

termed *improper*. In the latter case $\mathbf{L}$ can at best be written as a Jordan block:

$$\mathbf{L} = \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix}.$$

- $\det(\mathbf{L}) > 0$ and $4\det(\mathbf{L}) > \mathrm{tr}^2(\mathbf{L})$ (negative discriminant): $s_1 = s_2^*$ are complex conjugates with real part $\mathrm{tr}(\mathbf{L})/2$ and nonzero imaginary part; the trajectories are spirals and the fixed point is a *focus*, attractive (stable) if $\mathrm{tr}(\mathbf{L}) < 0$ or repulsive (unstable) if $\mathrm{tr}(\mathbf{L}) > 0$ (Figure 1.2c).
- $\det(\mathbf{L}) > 0$ and $\mathrm{tr}(\mathbf{L}) = 0$: $s_1 = s_2^*$ are purely imaginary; the trajectories are ellipses and the fixed point is a *center* (Figure 1.2d). A perturbation neither grows nor decays, and the stability is termed neutral.
- $\det(\mathbf{L}) = 0$: $\mathbf{L}$ is not invertible (Figure 1.2e). If $\mathrm{tr}(\mathbf{L}) \neq 0$, zero is a simple eigenvalue, whereas if $\mathrm{tr}(\mathbf{L}) = 0$, zero is a double eigenvalue. In the latter case, if the proper subspace has dimension 2, $\mathbf{L}$ is diagonalizable ($\mathbf{L} = \mathbf{0}$); otherwise $\mathbf{L}$ is a Jordan block of the form

$$\mathbf{L} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In the first three cases the real part of each of the two eigenvalues is *nonzero* and the fixed point is termed *hyperbolic*. In the last two cases the real parts are zero and the fixed point is termed *nonhyperbolic*.

As an example, let us consider the stability of the fixed point $(0, 0)$ of the system (1.3). The linearized system is written as

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = -\mu x_2 - \omega_0^2 x_1. \tag{1.6}$$

The trace and the determinant of the matrix of this system are respectively $-\mu$ and ω_0^2. The eigenvalues are $s_\pm = \frac{1}{2}(-\mu \pm \sqrt{\mu^2 - 4\omega_0^2})$. For $\mu < -2\omega_0$ or $\mu > 2\omega_0$ the discriminant is positive and the eigenvalues are real and of the same sign, that of $-\mu$; the fixed point is a node and determination of the eigenvectors permits the local phase portrait to be sketched. For $-2\omega_0 < \mu < 2\omega_0$ the eigenvalues are complex conjugates of each other, and the fixed point is a focus or a center for $\mu = 0$. In the end, $(0, 0)$ is attractive (stable) for $\mu > 0$ and repulsive (unstable) for $\mu < 0$. A similar analysis can be performed for the other fixed point $(\pi, 0)$, for which the trace and the determinant of the matrix $\mathbf{L}$ are respectively $-\mu$ and $-\omega_0^2$. The eigenvalues are real and of opposite signs, and so the fixed point is a saddle.

1.2.3 Stability of a nonhyperbolic fixed point

A special situation occurs when all the eigenvalues have negative real part except for one (or several) which have *zero* real part. The fixed point is then *nonhyperbolic*, and we can learn nothing about its stability from the linear stability analysis. Its stability is therefore determined by the nonlinear terms, whose effect can be stabilizing or destabilizing. Let us take as an example the oscillator described by the system (1.3) in the nondissipative case ($\mu = 0$) with an additional force $\beta(\mathrm{d}\theta/\mathrm{d}t)^3$. The system linearized about the fixed point $(0, 0)$ possesses two purely imaginary eigenvalues $\pm\mathrm{i}\omega_0$, and so the linear stability analysis tells us nothing. However, in this particular case it can be shown simply, without linearization, that the fixed point is stable for $\beta > 0$ and unstable for $\beta < 0$. We multiply the first equation in (1.3) by x_1 and the second by x_2 and then add them. Introducing the distance to the fixed point $r = \sqrt{x_1^2 + x_2^2}$, we obtain

$$r\frac{\mathrm{d}r}{\mathrm{d}t} = -\beta x_2^4. \tag{1.7}$$

The distance r therefore varies monotonically with time, decreasing for $\beta > 0$ and increasing for $\beta < 0$, thus proving the result.

1.3 Bifurcations

1.3.1 Definition

The behavior of a physical system depends in general on a certain number of parameters, for example, the damping constant μ of the oscillator (1.3). An important question is the following: how does the system behave when one of these parameters is varied? The answer is that nothing much happens except when the parameter passes through certain values where the qualitative behavior of the system changes. Let us take the oscillator (1.3) as an example. As μ varies without changing sign, the oscillator remains unstable when μ is negative, and stable when μ is positive. However, when μ passes through the critical value $\mu_c = 0$, the stability of the equilibrium position changes. It is said that the oscillator undergoes a *bifurcation* at $\mu = \mu_c$. The general definition of a bifurcation of a fixed point is the following.

Definition 1.1 Let a dynamical system depend on a parameter μ and possess a fixed point $\mathbf{a}(\mu)$. This system undergoes a *bifurcation* of the fixed point for $\mu = \mu_c$ if for this value of the parameter the system linearized at the fixed point $\mathbf{a}$ admits an eigenvalue with zero real part, i.e., if the fixed point is nonhyperbolic.

The rest of this section is devoted to the study of three important bifurcations.

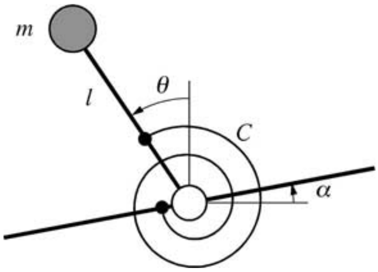

Figure 1.3 Schematic representation of the oscillator described by (1.10).

1.3.2 Saddle–node bifurcation

Let us consider the mechanical system represented in Figure 1.3. An arm of length l is attached to a pivot at its lower end and holds a mass m at its other end; its angular position is given by the angle θ. One end of a helical torsion spring with spring constant C is attached to the arm, while the other end of the spring is attached to a plane inclined at an angle α with respect to the horizontal. The spring tends to restore the arm to the direction perpendicular to the attached plane. We also include a moment of viscous friction $-mgl\tau^* \mathrm{d}\theta/\mathrm{d}t$ about the pivot, where τ^* is a relaxation time.

Denoting the mass, length, and time scales as m, l, and $\sqrt{l/g}$, the oscillator potential energy in the gravitational field can be written as

$$V(\omega^2, \alpha, \theta) = \frac{\omega^2}{2}(\theta - \alpha)^2 + \cos\theta - 1, \tag{1.8}$$

where the characteristic frequency ω is defined as

$$\omega^2 = \frac{C}{mgl}. \tag{1.9}$$

In terms of these scales the friction moment takes the form $-\tau \mathrm{d}\theta/\mathrm{d}t$, where $\tau = \tau^*/\sqrt{l/g}$ is the dimensionless relaxation time. The equation of motion is then

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \tau\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{\partial V}{\partial \theta}. \tag{1.10}$$

This equation can be rewritten as a dynamical system of two ODEs in the phase space $(\theta, \mathrm{d}\theta/\mathrm{d}t)$. The fixed points (equilibrium states) are defined by $\mathrm{d}\theta/\mathrm{d}t = 0$, and θ is the root of the equation for the potential extrema:

$$0 = \frac{\partial V}{\partial \theta} = \omega^2(\theta - \alpha) - \sin\theta. \tag{1.11}$$

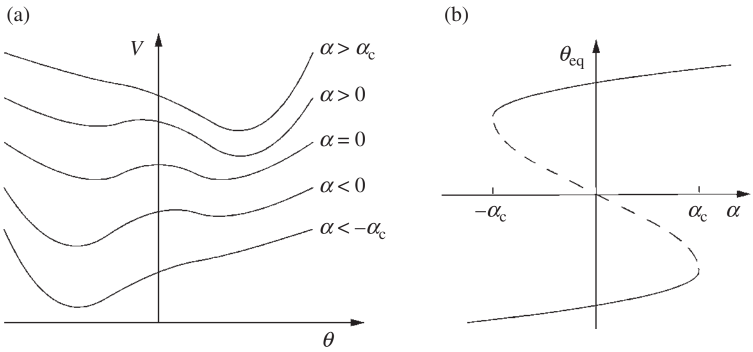

Figure 1.4 (a) The potential $V(\theta)$ for various inclinations α (the relative vertical positions of the various curves are arbitrary). (b) The bifurcation diagram: (—) stable states, (- -) unstable states.

The dependence of the equilibrium states on the two parameters ω^2 and α can be determined graphically or, for $|\alpha|$ small and ω^2 near unity, by a Taylor series expansion about $\theta = 0$. For $\alpha = 0$ the potentials at the equilibrium points θ_- and θ_+ are the same (Figure 1.4a). For $|\alpha|$ small and $\omega^2 < 1$ the system possesses an unstable equilibrium state θ_0 near $\theta = 0$ (the corresponding fixed point is a saddle) and two stable equilibrium states on either side, $\theta_- < 0$ and $\theta_+ > 0$ (whose corresponding fixed points are nodes). For $\alpha < 0$ the state $\theta_- < 0$ has the lowest potential and is therefore the most stable state, while the state $\theta_+ > 0$ is only metastable. The situation is reversed for $\alpha > 0$.

Let us consider the system in the state θ_- with α positive and small (Figure 1.4b). As α increases, the metastable equilibrium state θ_- and the unstable equilibrium state θ_0 approach each other, and there exists a critical inclination α_c for which the two equilibrium states merge. For $\alpha > \alpha_c$, the system jumps to the stable branch θ_+. For $\alpha = \alpha_c$, the phase portrait of the system therefore undergoes a qualitative change when the stable node $(\theta_-, 0)$ and the unstable saddle $(\theta_0, 0)$ coalesce. This qualitative change corresponds to a bifurcation: for $\alpha = \alpha_c$, an eigenvalue of the system linearized about each of the fixed points $(\theta_0, 0)$ and $(\theta_-, 0)$ crosses the imaginary axis (the proof is left as an exercise). The corresponding bifurcation is called a *saddle–node bifurcation*. A similar bifurcation occurs for decreasing α when α reaches the value $-\alpha_c$. Figure 1.4b, which shows the fixed points as a function of the parameter α, is called the *bifurcation diagram*. At each bifurcation the system jumps from one branch to another, and the critical value of the bifurcation parameter α is different depending on whether it is increasing or decreasing: the system displays hysteresis.

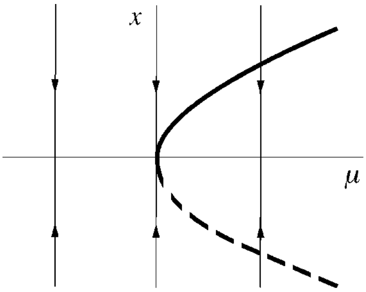

Figure 1.5 The saddle–node bifurcation diagram: (—) stable states, (- -) unstable states.

This example[2] displays a bifurcation corresponding to the coalescence of two fixed points, called a saddle–node bifurcation. The general definition of such a bifurcation is the following.

Definition 1.2 A dynamical system possessing a stable fixed point **a** undergoes a *saddle–node bifurcation* at $\mu = \mu_c$ if a real eigenvalue of the system linearized about **a** crosses the imaginary axis for $\mu = \mu_c$. For μ in the neighborhood of μ_c, the behavior of the system is then governed, maybe after an appropriate change of variables, by the following equation, called the normal form of a saddle–node bifurcation:

$$\frac{dx}{dt} = \mu - x^2. \tag{1.12}$$

Figure 1.5 shows the corresponding bifurcation diagram.

1.3.3 Pitchfork bifurcation

Let us return to the oscillator of Figure 1.3, and now consider what happens when we allow ω^2 to vary for fixed $\alpha = 0$. As ω^2 increases, the potential barrier between the two minima flattens, and the three equilibrium points coalesce for $\omega^2_{c0} = 1$ (Figure 1.6a). For $\omega^2 > \omega^2_{c0}$, only the stable equilibrium state $\theta = 0$ exists. This qualitative change of the phase portrait again corresponds to a bifurcation: for $\omega^2 = \omega^2_{c0}$, an eigenvalue of the system linearized about $(0, 0)$ crosses the imaginary axis (the proof is left as an exercise). The corresponding bifurcation is called a *supercritical pitchfork bifurcation* and the bifurcation diagram is shown in Figure 1.6b. The term supercritical means that in passing through the bifurcation a *stable* branch of equilibrium positions varies continuously, without any discontinuity.

[2] An extension of the analysis to the case of a chain of coupled oscillators can be found in Charru (1997).

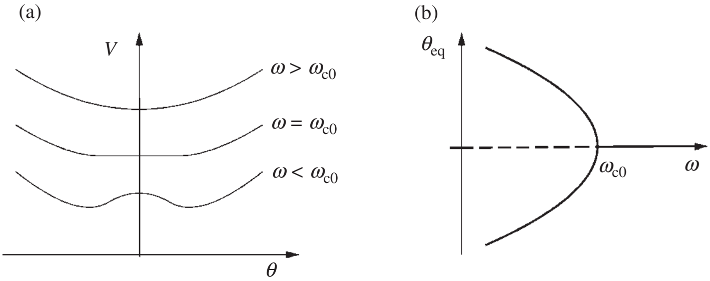

Figure 1.6 (a) The potential for various ω and $\alpha = 0$ (the relative vertical positions of the various curves are arbitrary). (b) Bifurcation diagram: (—) stable states, (- -) unstable states.

The existence of the pitchfork bifurcation displayed in this example is related in a crucial way to the symmetry of the problem about $\theta = 0$, i.e., to the invariance of the equation for the transformation of θ into $-\theta$, which is referred to as reflection invariance. A pitchfork bifurcation is defined more generally as follows.

Definition 1.3 A dynamical system which is invariant under reflection, i.e., invariant under the transformation $x \to -x$ (associated with a symmetry of the physical system), and which possesses a stable fixed point **a** undergoes a *pitchfork bifurcation* at $\mu = \mu_c$ if a real eigenvalue of the system linearized about **a** crosses the imaginary axis for $\mu = \mu_c$. For μ in the neighborhood of μ_c, the behavior of the system is then governed, perhaps after an appropriate change of variables, by the following equation, called the normal form of a pitchfork bifurcation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu x - \delta x^3, \qquad \delta = \pm 1. \tag{1.13}$$

The case $\delta = 1$ is termed *supercritical* and the case $\delta = -1$ is termed *subcritical*.

Figure 1.7 shows the corresponding bifurcation diagrams. In the supercritical case the equilibrium state $x = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$; in the latter case any perturbation of this equilibrium state makes the system jump to one of the stable branches $\pm\sqrt{\mu}$. In the subcritical case and for $\mu < 0$, $x = 0$ is always stable with respect to infinitesimal amplitude perturbations, but an amplitude perturbation larger than $\pm\sqrt{-\mu}$, i.e., a *perturbation of finite amplitude*, can destabilize it: for $\mu > 0$, any perturbation of the state $x = 0$ causes the system to jump discontinuously to a state that the normal form (1.13) is incapable of describing; higher-order terms (of degree five or higher) must be taken into account.

What happens in a system when the reflection symmetry $x \to -x$ is broken by an imperfection? We return to the oscillator of Figure 1.3 but now for small, nonzero angle α, which breaks the $\theta \to -\theta$ invariance, and we consider the effect