NIST Handbook of Mathematical Functions

Modern developments in theoretical and applied science depend on knowledge of the properties of mathematical functions, from elementary trigonometric functions to the multitude of special functions. These functions appear whenever natural phenomena are studied, engineering problems are formulated, and numerical simulations are performed. They also crop up in statistics, financial models, and economic analysis. Using them effectively requires practitioners to have ready access to a reliable collection of their properties.

This handbook results from a 10-year project conducted by the National Institute of Standards and Technology with an international group of expert authors and validators. Printed in full color, it is destined to replace its predecessor, the classic but long-outdated *Handbook of Mathematical Functions*, edited by Abramowitz and Stegun. Included with every copy of the book is a CD with a searchable PDF.

Frank W. J. Olver is Professor Emeritus in the Institute for Physical Science and Technology and the Department of Mathematics at the University of Maryland. From 1961 to 1986 he was a Mathematician at the National Bureau of Standards in Washington, D.C. Professor Olver has published 76 papers in refereed and leading mathematics journals, and he is the author of *Asymptotics and Special Functions* (1974). He has served as editor of *SIAM Journal on Numerical Analysis*, *SIAM Journal on Mathematical Analysis*, *Mathematics of Computation*, *Methods and Applications of Analysis*, and the *NBS Journal of Research*.

Daniel W. Lozier leads the Mathematical Software Group in the Mathematical and Computational Sciences Division of NIST. He received his Ph.D. in applied mathematics from the University of Maryland in 1979 and has been at NIST since 1970. He is an active member of the SIAM Activity Group on Orthogonal Polynomials and Special Functions, having served two terms as chair and one as vice-chair, and currently is serving as secretary. He has been an editor of *Mathematics of Computation* and the *NIST Journal of Research*.

Ronald F. Boisvert leads the Mathematical and Computational Sciences Division of the Information Technology Laboratory at NIST. He received his Ph.D. in computer science from Purdue University in 1979 and has been at NIST since then. He has served as editor-in-chief of the *ACM Transactions on Mathematical Software*. He is currently co-chair of the Publications Board of the Association for Computing Machinery (ACM) and chair of the International Federation for Information Processing (IFIP) Working Group 2.5 (Numerical Software).

Charles W. Clark received his Ph.D. in physics from the University of Chicago in 1979. He is a member of the U.S. Senior Executive Service and Chief of the Electron and Optical Physics Division and acting Group Leader of the NIST Synchrotron Ultraviolet Radiation Facility (SURF III). Clark serves as Program Manager for Atomic and Molecular Physics at the U.S. Office of Naval Research and is a Fellow of the Joint Quantum Institute of NIST and the University of Maryland at College Park and a Visiting Professor at the National University of Singapore.

Rainbow over Woolsthorpe Manor

From the frontispiece of the *Notes and Records of the Royal Society of London*, v. 36 (1981–82), with permission. Photograph by Dr. Roy L. Bishop, Physics Department, Acadia University, Nova Scotia, Canada, with permission.

Commentary

The faint line below the main colored arc is a *supernumerary rainbow*, produced by the interference of different sun-rays traversing a raindrop and emerging in the same direction. For each color, the intensity profile across the rainbow is an Airy function. Airy invented his function in 1838 precisely to describe this phenomenon more accurately than Young had done in 1800 when pointing out that supernumerary rainbows require the wave theory of light and are impossible to explain with Newton's picture of light as a stream of independent corpuscles. The house in the picture is Newton's birthplace.

Sir Michael V. Berry
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Foreword

In 1964 the National Institute of Standards and Technology[1] published the *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by Milton Abramowitz and Irene A. Stegun. That 1046-page tome proved to be an invaluable reference for the many scientists and engineers who use the special functions of applied mathematics in their day-to-day work, so much so that it became the most widely distributed and most highly cited NIST publication in the first 100 years of the institution's existence.[2] The success of the original handbook, widely referred to as "Abramowitz and Stegun" ("A&S"), derived not only from the fact that it provided critically useful scientific data in a highly accessible format, but also because it served to standardize definitions and notations for special functions. The provision of standard reference data of this type is a core function of NIST.

Much has changed in the years since A&S was published. Certainly, advances in applied mathematics have continued unabated. However, we have also seen the birth of a new age of computing technology, which has not only changed how we utilize special functions, but also how we communicate technical information. The document you are now holding, or the Web page you are now reading, represents an effort to extend the legacy of A&S well into the 21st century. The new printed volume, the *NIST Handbook of Mathematical Functions*, serves a similar function as the original A&S, though it is heavily updated and extended. The online version, the *NIST Digital Library of Mathematical Functions (DLMF)*, presents the same technical information along with extensions and innovative interactive features consistent with the new medium. The DLMF may well serve as a model for the effective presentation of highly mathematical reference material on the Web.

The production of these new resources has been a very complex undertaking some 10 years in the making. This could not have been done without the cooperation of many mathematicians, information technologists, and physical scientists both within NIST and externally. Their unfailing dedication is acknowledged deeply and gratefully. Particular attention is called to the generous support of the National Science Foundation, which made possible the participation of experts from academia and research institutes worldwide.

Dr. Patrick D. Gallagher
Director, NIST
November 20, 2009
Gaithersburg, Maryland

[1] Then known as the National Bureau of Standards.
[2] D. R. Lide (ed.), *A Century of Excellence in Measurement, Standards, and Technology*, CRC Press, 2001.

Preface

The *NIST Handbook of Mathematical Functions*, together with its Web counterpart, the *NIST Digital Library of Mathematical Functions (DLMF)*, is the culmination of a project that was conceived in 1996 at the National Institute of Standards and Technology (NIST). The project had two equally important goals: to develop an authoritative replacement for the highly successful *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, published in 1964 by the National Bureau of Standards (M. Abramowitz and I. A. Stegun, editors); and to disseminate essentially the same information from a public Web site operated by NIST. The new Handbook and DLMF are the work of many hands: editors, associate editors, authors, validators, and numerous technical experts. A summary of the responsibilities of these groups may help in understanding the structure and results of this project.

Executive responsibility was vested in the editors: Frank W. J. Olver (University of Maryland, College Park, and NIST), Daniel W. Lozier (NIST), Ronald F. Boisvert (NIST), and Charles W. Clark (NIST). Olver was responsible for organizing and editing the mathematical content after receiving it from the authors; for communicating with the associate editors, authors, validators, and other technical experts; and for assembling the **Notations** section and the **Index**. In addition, Olver was author or co-author of five chapters. Lozier directed the NIST research, technical, and support staff associated with the project, administered grants and contracts, together with Boisvert compiled the **Software** sections for the Web version of the chapters, conducted editorial and staff meetings, represented the project within NIST and at professional meetings in the United States and abroad, and together with Olver carried out the day-to-day development of the project. Boisvert and Clark were responsible for advising and assisting in matters related to the use of information technology and applications of special functions in the physical sciences (and elsewhere); they also participated in the resolution of major administrative problems when they arose.

The associate editors are eminent domain experts who were recruited to advise the project on strategy, execution, subject content, format, and presentation, and to help identify and recruit suitable candidate authors and validators. The associate editors were:

Richard A. Askey
University of Wisconsin, Madison

Michael V. Berry
University of Bristol

Walter Gautschi (resigned 2002)
Purdue University

Leonard C. Maximon
George Washington University

Morris Newman
University of California, Santa Barbara

Ingram Olkin
Stanford University

Peter Paule
Johannes Kepler University

William P. Reinhardt
University of Washington

Nico M. Temme
Centrum voor Wiskunde en Informatica

Jet Wimp (resigned 2001)
Drexel University

The technical information provided in the Handbook and DLMF was prepared by subject experts from around the world. They are identified on the title pages of the chapters for which they served as authors and in the table of Contents.

The validators played a critical role in the project, one that was absent in its 1964 counterpart: to provide critical, independent reviews during the development of each chapter, with attention to accuracy and appropriateness of subject coverage. These reviews have contributed greatly to the quality of the product. The validators were:

T. M. Apostol
California Institute of Technology

A. R. Barnett
University of Waikato, New Zealand

A. I. Bobenko
Technische Universität, Berlin

B. B. L. Braaksma
University of Groningen

D. M. Bressoud
Macalester College

B. C. Carlson
Iowa State University

B. Deconinck
University of Washington

T. M. Dunster
University of California, San Diego

A. Gil
Universidad de Cantabria

A. R. Its
Indiana University–Purdue University, Indianapolis

B. R. Judd
Johns Hopkins University

R. Koekoek
Delft University of Technology

T. H. Koornwinder
University of Amsterdam

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Pfizer Global R&D

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University of Wisconsin, Milwaukee

G. Wolf
Universität Duisberg-Essen

R. Wong
City University of Hong Kong

All of the mathematical information contained in the Handbook is also contained in the DLMF, along with additional features such as more graphics, expanded tables, and higher members of some families of formulas; in consequence, in the Handbook there are occasional gaps in the numbering sequences of equations, tables, and figures. The Web address where additional DLMF content can be found is printed in blue at appropriate places in the Handbook. The home page of the DLMF is accessible at `http://dlmf.nist.gov/`.

The DLMF has been constructed specifically for effective Web usage and contains features unique to Web presentation. The Web pages contain many active links, for example, to the definitions of symbols within the DLMF, and to external sources of reviews, full texts of articles, and items of mathematical software. Advanced capabilities have been developed at NIST for the DLMF, and also as part of a larger research effort intended to promote the use of the Web as a tool for doing mathematics. Among these capabilities are: a facility to allow users to download LaTeX and MathML encodings of every formula into document processors and software packages (eventually, a fully semantic downloading capability may be possible); a search engine that allows users to locate formulas based on queries expressed in mathematical notation; and user-manipulable 3-dimensional color graphics.

Production of the Handbook and DLMF was a mammoth undertaking, made possible by the dedicated leadership of Bruce R. Miller (NIST), Bonita V. Saunders (NIST), and Abdou S. Youssef (George Washington University and NIST). Miller was responsible for information architecture, specializing LaTeX for the needs of the project, translation from LaTeX to MathML, and the search interface. Saunders was responsible for mesh generation for curves and surfaces, data computation and validation, graphics production, and interactive Web visualization. Youssef was responsible for mathematics search indexing and query processing. They were assisted by the following NIST staff: Marjorie A. McClain (LaTeX, bibliography), Joyce E. Conlon (bibliography), Gloria Wiersma (LaTeX), Qiming Wang (graphics generation, graphics viewers), and Brian Antonishek (graphics viewers).

The editors acknowledge the many other individuals who contributed to the project in a variety of ways. Among the research, technical, and support staff at NIST these are B. K. Alpert, T. M. G. Arrington, R. Bickel, B. Blaser, P. T. Boggs, S. Burley, G. Chu, A. Dienstfrey, M. J. Donahue, K. R. Eberhardt, B. R. Fabijonas, M. Fancher, S. Fletcher, J. Fowler, S. P. Frechette, C. M. Furlani, K. B. Gebbie, C. R. Hagwood, A. N. Heckert, M. Huber, P. K. Janert, R. N. Kacker, R. F. Kayser, P. M. Ketcham, E. Kim, M. J. Lieber-

man, R. R. Lipman, M. S. Madsen, E. A. P. Mai, W. Mehuron, P. J. Mohr, S. Olver, D. R. Penn, S. Phoha, A. Possolo, S. P. Ressler, M. Rubin, J. Rumble, C. A. Schanzle, B. I. Schneider, N. Sedransk, E. L. Shirley, G. W. Stewart, C. P. Sturrock, G. Thakur, S. Wakid, and S. F. Zevin. Individuals from outside NIST are S. S. Antman, A. M. Ashton, C. M. Bender, J. J. Benedetto, R. L. Bishop, J. M. Borwein, H. W. Braden, C. Brezinski, F. Chyzak, J. N. L. Connor, R. Cools, A. Cuyt, I. Daubechies, P. J. Davis, C. F. Dunkl, J. P. Goedbloed, B. Gordon, J. W. Jenkins, L. H. Kellogg, C. D. Kemp, K. S. Kölbig, S. G. Krantz, M. D. Kruskal, W. Lay, D. A. Lutz, E. L. Mansfield, G. Marsaglia, B. M. McCoy, W. Miller, Jr., M. E. Muldoon, S. P. Novikov, P. J. Olver, W. C. Parke, M. Petkovsek, W. H. Reid, B. Salvy, C. Schneider, M. J. Seaton, N. C. Severo, I. A. Stegun, F. Stenger, M. Steuerwalt, W. G. Strang, P. R. Turner, J. Van Deun, M. Vuorinen, E. J. Weniger, H. Wiersma, R. C. Winther, D. B. Zagier, and M. Zelen. Undoubtedly, the editors have overlooked some individuals who contributed, as is inevitable in a large long-lasting project. Any oversight is unintentional, and the editors apologize in advance.

The project was funded in part by NSF Award 9980036, administered by the NSF's Knowledge and Distributed Intelligence Program. Within NIST financial resources and staff were committed by the Information Technology Laboratory, Physics Laboratory, Systems Integration for Manufacturing Applications Program of the Manufacturing Engineering Laboratory, Standard Reference Data Program, and Advanced Technology Program.

Notwithstanding the great care that has been exercised by the editors, authors, validators, and the NIST staff, it is almost inevitable that in a work of the magnitude and scope of the NIST Handbook and DLMF errors will still be present. Users need to be aware that none of these individuals nor the National Institute of Standards and Technology can assume responsibility for any possible consequences of such errors.

Lastly, the editors appreciate the skill, and long experience, that was brought to bear by the publisher, Cambridge University Press, on the production and publication of the new Handbook.

Frank W. J. Olver
Editor-in-Chief and Mathematics Editor

Daniel W. Lozier
General Editor

Ronald F. Boisvert
Information Technology Editor

Charles W. Clark
Physical Sciences Editor

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Mathematical Introduction

Organization and Objective

The mathematical content of the *NIST Handbook of Mathematical Functions* has been produced over a ten-year period. This part of the project has been carried out by a team comprising the mathematics editor, authors, validators, and the NIST professional staff. Also, valuable initial advice on all aspects of the project was provided by ten external associate editors.

The NIST Handbook has essentially the same objective as the *Handbook of Mathematical Functions* that was issued in 1964 by the National Bureau of Standards as Number 55 in the NBS Applied Mathematics Series (AMS). This objective is to provide a reference tool for researchers and other users in applied mathematics, the physical sciences, engineering, and elsewhere who encounter special functions in the course of their everyday work.

The mathematical project team has endeavored to take into account the hundreds of research papers and numerous books on special functions that have appeared since 1964. As a consequence, in addition to providing more information about the special functions that were covered in AMS 55, the NIST Handbook includes several special functions that have appeared in the interim in applied mathematics, the physical sciences, and engineering, as well as in other areas. See, for example, Chapters 16, 17, 18, 19, 21, 27, 29, 31, 32, 34, 35, and 36.

Two other ways in which this Handbook differs from AMS 55, and other handbooks, are as follows.

First, the editors instituted a validation process for the whole technical content of each chapter. This process greatly extended normal editorial checking procedures. All chapters went through several drafts (nine in some cases) before the authors, validators, and editors were fully satisfied.

Secondly, as described in the **Preface**, a Web version (the NIST DLMF) is also available.

Methodology

The first three chapters of the NIST Handbook and DLMF are methodology chapters that provide detailed coverage of, and references for, mathematical topics that are especially important in the theory, computation, and application of special functions. (These chapters can also serve as background material for university graduate courses in complex variables, classical analysis, and numerical analysis.)

Particular care is taken with topics that are not dealt with sufficiently thoroughly from the standpoint of this Handbook in the available literature. These include, for example, multivalued functions of complex variables, for which new definitions of branch points and principal values are supplied (§§1.10(vi), 4.2(i)); the Dirac delta (or delta function), which is introduced in a more readily comprehensible way for mathematicians (§1.17); numerically satisfactory solutions of differential and difference equations (§§2.7(iv), 2.9(i)); and numerical analysis for complex variables (Chapter 3).

In addition, there is a comprehensive account of the great variety of analytical methods that are used for deriving and applying the extremely important asymptotic properties of the special functions, including double asymptotic properties (Chapter 2 and §§10.41(iv), 10.41(v)).

Notation for the Special Functions

The first section in each of the special function chapters (Chapters 5–36) lists notation that has been adopted for the functions in that chapter. This section may also include important alternative notations that have appeared in the literature. With a few exceptions the adopted notations are the same as those in standard applied mathematics and physics literature.

The exceptions are ones for which the existing notations have drawbacks. For example, for the hypergeometric function we often use the notation $\mathbf{F}(a,b;c;z)$ (§15.2(i)) in place of the more conventional ${}_2F_1(a,b;c;z)$ or $F(a,b;c;z)$. This is because $\mathbf{F}$ is akin to the notation used for Bessel functions (§10.2(ii)), inasmuch as $\mathbf{F}$ is an entire function of each of its parameters a, b, and c: this results in fewer restrictions and simpler equations. Similarly in the case of confluent hypergeometric functions (§13.2(i)).

Other examples are: (a) the notation for the Ferrers functions—also known as associated Legendre functions on the cut—for which existing notations can easily be confused with those for other associated Legendre functions (§14.1); (b) the spherical Bessel functions for which existing notations are unsymmetric and inelegant (§§10.47(i) and 10.47(ii)); and (c) elliptic integrals for which both Legendre's forms and the more recent symmetric forms are treated fully (Chapter 19).

The **Notations** section beginning on p. 873 includes all the notations for the special functions adopted in this Handbook. In the corresponding section for the DLMF some of the alternative notations that appear in the first section of the special function chapters are also included.

Common Notations and Definitions

$\mathbb{C}$	complex plane (excluding infinity).
D	decimal places.
det	determinant.
$\delta_{j,k}$ or δ_{jk}	Kronecker delta: 0 if $j \neq k$; 1 if $j = k$.
Δ (or Δ_x)	forward difference operator: $\Delta f(x) = f(x+1) - f(x)$.
∇ (or ∇_x)	backward difference operator: $\nabla f(x) = f(x) - f(x-1)$. (See also del operator in the **Notations** section.)
empty sums	zero.
empty products	unity.
$\in$	element of.
$\notin$	not an element of.
$\forall$	for every.
$\Longrightarrow$	implies.
$\Longleftrightarrow$	is equivalent to.
$n!$	factorial: $1 \cdot 2 \cdot 3 \cdots n$ if $n = 1, 2, 3, \ldots$; 1 if $n = 0$.
$n!!$	double factorial: $2 \cdot 4 \cdot 6 \cdots n$ if $n = 2, 4, 6, \ldots$; $1 \cdot 3 \cdot 5 \cdots n$ if $n = 1, 3, 5, \ldots$; 1 if $n = 0, -1$.
$\lfloor x \rfloor$	floor or integer part: the integer such that $x - 1 < \lfloor x \rfloor \leq x$, with x real.
$\lceil x \rceil$	ceiling: the integer such that $x \leq \lceil x \rceil < x + 1$, with x real.
$f(z)\vert_C = 0$	$f(z)$ is continuous at all points of a simple closed contour C in $\mathbb{C}$.
$< \infty$	is finite, or converges.
$\gg$	much greater than.
$\Im$	imaginary part.
iff	if and only if.
inf	greatest lower bound (infimum).
sup	least upper bound (supremum).
$\cap$	intersection.
$\cup$	union.
(a, b)	open interval in $\mathbb{R}$, or open straight-line segment joining a and b in $\mathbb{C}$.
$[a, b]$	closed interval in $\mathbb{R}$, or closed straight-line segment joining a and b in $\mathbb{C}$.
$(a, b]$ or $[a, b)$	half-closed intervals.
$\subset$	is contained in.
$\subseteq$	is, or is contained in.
lim inf	least limit point.
$[a_{j,k}]$ or $[a_{jk}]$	matrix with (j,k)th element $a_{j,k}$ or a_{jk}.
$\mathbf{A}^{-1}$	inverse of matrix $\mathbf{A}$.
$\operatorname{tr} \mathbf{A}$	trace of matrix $\mathbf{A}$.
$\mathbf{A}^{\mathrm{T}}$	transpose of matrix $\mathbf{A}$.
$\mathbf{I}$	unit matrix.
mod or modulo	$m \equiv n \pmod{p}$ means p divides $m - n$, where m, n, and p are positive integers with $m > n$.
$\mathbb{N}$	set of all positive integers.
$(\alpha)_n$	Pochhammer's symbol: $\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$ if $n = 1, 2, 3, \ldots$; 1 if $n = 0$.
$\mathbb{Q}$	set of all rational numbers.
$\mathbb{R}$	real line (excluding infinity).
$\Re$	real part.
res	residue.
S	significant figures.
$\operatorname{sign} x$	-1 if $x < 0$; 0 if $x = 0$; 1 if $x > 0$.
$\backslash$	set subtraction.
$\mathbb{Z}$	set of all integers.
$n\mathbb{Z}$	set of all integer multiples of n.

Graphics

Special functions with one real variable are depicted graphically with conventional two-dimensional (2D) line graphs. See, for example, Figures 10.3.1–10.3.4.

With two real variables, special functions are depicted as 3D surfaces, with vertical height corresponding to the value of the function, and coloring added to emphasize the 3D nature. See Figures 10.3.5–10.3.8 for examples.

Special functions with a complex variable are depicted as colored 3D surfaces in a similar way to functions of two real variables, but with the vertical height corresponding to the modulus (absolute value) of the function. See, for example, Figures 5.3.4–5.3.6. However, in many cases the coloring of the surface is chosen instead to indicate the quadrant of the plane to which the phase of the function belongs, thereby achieving a 4D effect. In these cases the phase colors that correspond to the 1st, 2nd, 3rd, and 4th quadrants are arranged in alphabetical order: blue, green, red, and yellow, respectively, and a “Quadrant Colors” icon appears alongside the figure. See, for example, Figures 10.3.9–10.3.16.

Lastly, users may notice some lack of smoothness in the color boundaries of some of the 4D-type surfaces; see, for example, Figure 10.3.9. This nonsmoothness arises because the mesh that was used to generate the figure was optimized only for smoothness of the surface, and not for smoothness of the color boundaries.

Applications

All of the special function chapters include sections devoted to mathematical, physical, and sometimes other applications of the main functions in the chapter. The purpose of these sections is simply to illustrate the importance of the functions in other disciplines; no attempt is made to provide exhaustive coverage.

Computation

All of the special function chapters contain sections that describe available methods for computing the main functions in the chapter, and most also provide references to numerical tables of, and approximations for, these functions. In addition, the DLMF provides references to research papers in which software is developed, together with links to sites where the software can be obtained.

In referring to the numerical tables and approximations we use notation typified by $x = 0(.05)1$, 8D or 8S. This means that the variable x ranges from 0 to 1 in intervals of 0.05, and the corresponding function values are tabulated to 8 decimal places or 8 significant figures.

Another numerical convention is that decimals followed by dots are unrounded; without the dots they are rounded. For example, to 4D π is $3.1415\ldots$ (unrounded) and 3.1416 (rounded).

Verification

For all equations and other technical information this Handbook and the DLMF either provide references to the literature for proof or describe steps that can be followed to construct a proof. In the Handbook this information is grouped at the section level and appears under the heading **Sources** in the **References** section. In the DLMF this information is provided in pop-up windows at the subsection level.

For equations or other technical information that appeared previously in AMS 55, the DLMF usually includes the corresponding AMS 55 equation number, or other form of reference, together with corrections, if needed. However, none of these citations are to be regarded as supplying proofs.

Special Acknowledgment

I pay tribute to my friend and predecessor Milton Abramowitz. His genius in the creation of the *National Bureau of Standards Handbook of Mathematical Functions* paid enormous dividends to the world's scientific, mathematical, and engineering communities, and paved the way for the development of the *NIST Handbook of Mathematical Functions* and *NIST Digital Library of Mathematical Functions*.

Frank W. J. Olver, *Mathematics Editor*