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Chapter 1

Algebraic and Analytic Methods

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Notation

1.1 Special Notation

(For other notation see pp. xiv and 873.)

x, y	real variables.
z	real variable in §§1.5–1.6.
z, w	complex variables in §§1.9–1.11.
j, k, ℓ	integers.
m, n	nonnegative integers, unless specified otherwise.
$\langle f, g\rangle$	distribution.
deg	degree.
primes	derivatives with respect to the variable, except where indicated otherwise.

Areas

1.2 Elementary Algebra

1.2(i) Binomial Coefficients

In (1.2.1)–(1.2.5) k and n are nonnegative integers and $k \le n$.

1.2.1 $$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}.$$

Binomial Theorem

1.2.2 $$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n.$$

1.2.3 $$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

1.2.4 $$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n} = 0.$$

1.2.5 $$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{k} = 2^{n-1},$$

where k is n or $n-1$ according as n is even or odd.

In (1.2.6)–(1.2.9) k and m are nonnegative integers and n is unrestricted.

1.2.6 $$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{(-1)^k(-n)_k}{k!} = (-1)^k\binom{k-n-1}{k}.$$

1.2.7 $$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

1.2.8 $$\sum_{k=0}^{m}\binom{n+k}{k} = \binom{n+m+1}{m}.$$

1.2.9 $$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^m\binom{n}{m} = (-1)^m\binom{n-1}{m}.$$

1.2(ii) Finite Series

Arithmetic Progression

1.2.10 $$a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) = na + \tfrac{1}{2}n(n-1)d = \tfrac{1}{2}n(a+\ell),$$

where ℓ = last term of the series $= a + (n-1)d$.

Geometric Progression

1.2.11 $$a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1-x^n)}{1-x}, \qquad x \ne 1.$$

1.2(iii) Partial Fractions

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct constants, and $f(x)$ be a polynomial of degree less than n. Then

1.2.12 $$\frac{f(x)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \cdots + \frac{A_n}{x-\alpha_n},$$

where

1.2.13 $$A_j = \frac{f(\alpha_j)}{\prod_{k\ne j}(\alpha_j - \alpha_k)}.$$

Also,

1.2.14 $$\frac{f(x)}{(x-\alpha_1)^n} = \frac{B_1}{x-\alpha_1} + \frac{B_2}{(x-\alpha_1)^2} + \cdots + \frac{B_n}{(x-\alpha_1)^n},$$

where

1.2.15 $$B_j = \frac{f^{(n-j)}(\alpha_1)}{(n-j)!},$$

and $f^{(k)}$ is the k-th derivative of f (§1.4(iii)).

If $m_1, m_2, \ldots, m_n$ are positive integers and $\deg f < \sum_{j=1}^{n} m_j$, then there exist polynomials $f_j(x)$, $\deg f_j < m_j$, such that

1.2.16 $$\frac{f(x)}{(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_n)^{m_n}} = \frac{f_1(x)}{(x-\alpha_1)^{m_1}} + \frac{f_2(x)}{(x-\alpha_2)^{m_2}} + \cdots + \frac{f_n(x)}{(x-\alpha_n)^{m_n}}.$$

To find the polynomials $f_j(x)$, $j = 1, 2, \ldots, n$, multiply both sides by the denominator of the left-hand side and equate coefficients. See Chrystal (1959, pp. 151–159).

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1.2(iv) Means

The *arithmetic mean* of n numbers $a_1, a_2, \dots, a_n$ is

1.2.17 $$A = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

The *geometric mean* G and *harmonic mean* H of n positive numbers $a_1, a_2, \dots, a_n$ are given by

1.2.18 $$G = (a_1 a_2 \cdots a_n)^{1/n},$$

1.2.19 $$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

If r is a nonzero real number, then the *weighted mean* $M(r)$ of n nonnegative numbers $a_1, a_2, \dots, a_n$, and n positive numbers $p_1, p_2, \dots, p_n$ with

1.2.20 $$p_1 + p_2 + \cdots + p_n = 1,$$

is defined by

1.2.21 $$M(r) = (p_1 a_1^r + p_2 a_2^r + \cdots + p_n a_n^r)^{1/r},$$

with the exception

1.2.22 $$M(r) = 0, \quad r < 0 \text{ and } a_1 a_2 \dots a_n = 0.$$

1.2.23 $$\lim_{r\to\infty} M(r) = \max(a_1, a_2, \dots, a_n),$$

1.2.24 $$\lim_{r\to-\infty} M(r) = \min(a_1, a_2, \dots, a_n).$$

For $p_j = 1/n$, $j = 1, 2, \dots, n$,

1.2.25 $$M(1) = A, \quad M(-1) = H,$$

and

1.2.26 $$\lim_{r\to 0} M(r) = G.$$

The last two equations require $a_j > 0$ for all j.

1.3 Determinants

1.3(i) Definitions and Elementary Properties

1.3.1 $$\det[a_{jk}] = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

1.3.2
$$\begin{aligned}\det[a_{jk}] &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.\end{aligned}$$

Higher-order determinants are natural generalizations. The *minor* M_{jk} of the entry a_{jk} in the nth-order determinant $\det[a_{jk}]$ is the $(n-1)$th-order determinant derived from $\det[a_{jk}]$ by deleting the jth row and the kth column. The *cofactor* A_{jk} of a_{jk} is

1.3.3 $$A_{jk} = (-1)^{j+k} M_{jk}.$$

An nth-order determinant expanded by its jth row is given by

1.3.4 $$\det[a_{jk}] = \sum_{\ell=1}^{n} a_{j\ell} A_{j\ell}.$$

If two rows (or columns) of a determinant are interchanged, then the determinant changes sign. If two rows (columns) of a determinant are identical, then the determinant is zero. If all the elements of a row (column) of a determinant are multiplied by an arbitrary factor μ, then the result is a determinant which is μ times the original. If μ times a row (column) of a determinant is added to another row (column), then the value of the determinant is unchanged.

1.3.5 $$\det[a_{jk}]^{\mathrm{T}} = \det[a_{jk}],$$

1.3.6 $$\det[a_{jk}]^{-1} = \frac{1}{\det[a_{jk}]},$$

1.3.7 $$\det([a_{jk}][b_{jk}]) = (\det[a_{jk}])(\det[b_{jk}]).$$

Hadamard's Inequality

For real-valued a_{jk},

1.3.8 $$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^2 \le (a_{11}^2 + a_{12}^2)(a_{21}^2 + a_{22}^2),$$

1.3.9 $$\det[a_{jk}]^2 \le \left(\sum_{k=1}^{n} a_{1k}^2\right)\left(\sum_{k=1}^{n} a_{2k}^2\right)\cdots\left(\sum_{k=1}^{n} a_{nk}^2\right).$$

Compare also (1.3.7) for the left-hand side. Equality holds iff

1.3.10 $$a_{j1}a_{k1} + a_{j2}a_{k2} + \cdots + a_{jn}a_{kn} = 0$$

for every distinct pair of j, k, or when one of the factors $\sum_{k=1}^{n} a_{jk}^2$ vanishes.

1.3(ii) Special Determinants

An *alternant* is a determinant function of n variables which changes sign when two of the variables are interchanged. Examples:

1.3.11 $$\det[f_k(x_j)], \qquad j = 1, \dots, n;\ k = 1, \dots, n,$$

1.3.12 $$\det[f(x_j, y_k)], \qquad j = 1, \dots, n;\ k = 1, \dots, n.$$

Vandermonde Determinant or Vandermondian

1.3.13 $$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1\le j<k\le n} (x_k - x_j).$$

Cauchy Determinant

1.3.14
$$\det\left[\frac{1}{a_j - b_k}\right] = (-1)^{n(n-1)/2} \times \prod_{1\le j<k\le n}(a_k - a_j)(b_k - b_j) \Big/ \prod_{j,k=1}^{n}(a_j - b_k).$$

Circulant

1.3.15
$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{vmatrix} = \prod_{k=1}^{n}(a_1 + a_2\omega_k + a_3\omega_k^2 + \cdots + a_n\omega_k^{n-1}),$$

where $\omega_1, \omega_2, \dots, \omega_n$ are the nth roots of unity (1.11.21).

Krattenthaler's Formula

For

1.3.16
$$t_{jk} = (x_j + a_n)(x_j + a_{n-1})\cdots(x_j + a_{k+1}) \times (x_j + b_k)(x_j + b_{k-1})\cdots(x_j + b_2),$$

1.3.17
$$\det[t_{jk}] = \prod_{1\le j<k\le n}(x_j - x_k)\prod_{2\le j\le k\le n}(b_j - a_k).$$

1.3(iii) Infinite Determinants

Let $a_{j,k}$ be defined for all integer values of j and k, and $D_n[a_{j,k}]$ denote the $(2n+1)\times(2n+1)$ determinant

1.3.18
$$D_n[a_{j,k}] = \begin{vmatrix} a_{-n,-n} & a_{-n,-n+1} & \cdots & a_{-n,n} \\ a_{-n+1,-n} & a_{-n+1,-n+1} & \cdots & a_{-n+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,-n} & a_{n,-n+1} & \cdots & a_{n,n} \end{vmatrix}.$$

If $D_n[a_{j,k}]$ tends to a limit L as $n \to \infty$, then we say that the *infinite determinant* $D_\infty[a_{j,k}]$ *converges* and $D_\infty[a_{j,k}] = L$.

Of importance for special functions are infinite determinants of *Hill's type*. These have the property that the double series

1.3.19
$$\sum_{j,k=-\infty}^{\infty} |a_{j,k} - \delta_{j,k}|$$

converges (§1.9(vii)). Here $\delta_{j,k}$ is the Kronecker delta. Hill-type determinants always converge.

For further information see Whittaker and Watson (1927, pp. 36–40) and Magnus and Winkler (1966, §2.3).

1.4 Calculus of One Variable

1.4(i) Monotonicity

If $f(x_1) \le f(x_2)$ for every pair x_1, x_2 in an interval I such that $x_1 < x_2$, then $f(x)$ is *nondecreasing* on I. If the $\le$ sign is replaced by $<$, then $f(x)$ is *increasing* (also called *strictly increasing*) on I. Similarly for *nonincreasing* and *decreasing* (*strictly decreasing*) functions. Each of the preceding four cases is classified as *monotonic*; sometimes *strictly monotonic* is used for the strictly increasing or strictly decreasing cases.

1.4(ii) Continuity

A function $f(x)$ is *continuous on the right* (or *from above*) at $x = c$ if

1.4.1
$$f(c+) \equiv \lim_{x\to c+} f(x) = f(c),$$

that is, for every arbitrarily small positive constant ϵ there exists δ (> 0) such that

1.4.2
$$|f(c+\alpha) - f(c)| < \epsilon,$$

for all α such that $0 \le \alpha < \delta$. Similarly, it is *continuous on the left* (or *from below*) at $x = c$ if

1.4.3
$$f(c-) \equiv \lim_{x\to c-} f(x) = f(c).$$

And $f(x)$ is *continuous at* c when both (1.4.1) and (1.4.3) apply.

If $f(x)$ is continuous at each point $c \in (a,b)$, then $f(x)$ is *continuous on the interval* (a,b) and we write $f \in C(a,b)$. If also $f(x)$ is continuous on the right at $x = a$, and continuous on the left at $x = b$, then $f(x)$ is *continuous on the interval* $[a,b]$, and we write $f(x) \in C[a,b]$.

A *removable singularity* of $f(x)$ at $x = c$ occurs when $f(c+) = f(c-)$ but $f(c)$ is undefined. For example, $f(x) = (\sin x)/x$ with $c = 0$.

A *simple discontinuity* of $f(x)$ at $x = c$ occurs when $f(c+)$ and $f(c-)$ exist, but $f(c+) \ne f(c-)$. If $f(x)$ is continuous on an interval I save for a finite number of simple discontinuities, then $f(x)$ is *piecewise* (or *sectionally*) continuous on I. For an example, see Figure 1.4.1

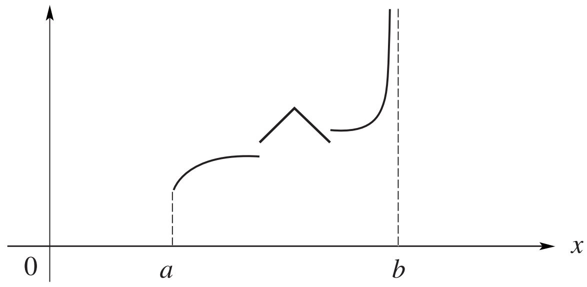

Figure 1.4.1: Piecewise continuous function on $[a,b)$.

1.4(iii) Derivatives

The *derivative* $f'(x)$ of $f(x)$ is defined by

1.4.4 $$f'(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

When this limit exists f is *differentiable* at x.

1.4.5 $$(f+g)'(x) = f'(x) + g'(x),$$

1.4.6 $$(fg)'(x) = f'(x)g(x) + f(x)g'(x),$$

1.4.7 $$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Higher Derivatives

1.4.8 $$f^{(2)}(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right),$$

1.4.9 $$f^{(n)} = f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x).$$

If $f^{(n)}$ exists and is continuous on an interval I, then we write $f \in C^n(I)$. When $n \geq 1$, f is *continuously differentiable* on I. When n is unbounded, f is *infinitely differentiable* on I and we write $f \in C^\infty(I)$.

Chain Rule

For $h(x) = f(g(x))$,

1.4.10 $$h'(x) = f'(g(x))g'(x).$$

Maxima and Minima

A necessary condition that a differentiable function $f(x)$ has a *local maximum* (*minimum*) at $x = c$, that is, $f(x) \leq f(c)$, ($f(x) \geq f(c)$) in a *neighborhood* $c - \delta \leq x \leq c + \delta$ ($\delta > 0$) of c, is $f'(c) = 0$.

Mean Value Theorem

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ such that

1.4.11 $$f(b) - f(a) = (b-a)f'(c).$$

If $f'(x) \geq 0$ (≤ 0) ($= 0$) for all $x \in (a,b)$, then f is nondecreasing (nonincreasing) (constant) on (a,b).

Leibniz's Formula

1.4.12 $$(fg)^{(n)} = f^{(n)}g + \binom{n}{1} f^{(n-1)}g' + \cdots + \binom{n}{k} f^{(n-k)}g^{(k)} + \cdots + fg^{(n)}.$$

Faà Di Bruno's Formula

1.4.13 $$\frac{d^n}{dx^n} f(g(x)) = \sum \left(\frac{n!}{m_1! m_2! \cdots m_n!}\right) f^{(k)}(g(x)) \times \left(\frac{g'(x)}{1!}\right)^{m_1} \left(\frac{g''(x)}{2!}\right)^{m_2} \cdots \left(\frac{g^{(n)}(x)}{n!}\right)^{m_n},$$

where the sum is over all nonnegative integers $m_1, m_2, \ldots, m_n$ that satisfy $m_1 + 2m_2 + \cdots + nm_n = n$, and $k = m_1 + m_2 + \cdots + m_n$.

L'Hôpital's Rule

If

1.4.14 $$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad (\text{or } \infty),$$

then

1.4.15 $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

when the last limit exists.

1.4(iv) Indefinite Integrals

If $F'(x) = f(x)$, then $\int f\,dx = F(x) + C$, where C is a constant.

Integration by Parts

1.4.16 $$\int fg\,dx = \left(\int f\,dx\right) g - \int \left(\int f\,dx\right) \frac{dg}{dx}\,dx.$$

1.4.17 $$\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C, & n \neq -1, \\ \ln|x| + C, & n = -1. \end{cases}$$

For the function ln see §4.2(i).

See §§4.10, 4.26(ii), 4.26(iv), 4.40(ii), and 4.40(iv) for indefinite integrals involving the elementary functions.

For extensive tables of integrals, see Apelblat (1983), Bierens de Haan (1867), Gradshteyn and Ryzhik (2000), Gröbner and Hofreiter (1949, 1950), and Prudnikov *et al.* (1986a,b, 1990, 1992a,b).

1.4(v) Definite Integrals

Suppose $f(x)$ is defined on $[a,b]$. Let $a = x_0 < x_1 < \cdots < x_n = b$, and ξ_j denote any point in $[x_j, x_{j+1}]$, $j = 0, 1, \ldots, n-1$. Then

1.4.18 $$\int_a^b f(x)\,dx = \lim \sum_{j=0}^{n-1} f(\xi_j)(x_{j+1} - x_j)$$

as $\max(x_{j+1} - x_j) \to 0$. Continuity, or piecewise continuity, of $f(x)$ on $[a,b]$ is sufficient for the limit to exist.

1.4.19 $$\int_a^b (cf(x) + dg(x))\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx,$$

c and d constants.

1.4.20 $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

1.4.21 $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

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Infinite Integrals

1.4.22 $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

Similarly for $\int_{-\infty}^a$. Next, if $f(b)=\pm\infty$, then

1.4.23 $$\int_a^b f(x)\,dx = \lim_{c\to b-}\int_a^c f(x)\,dx.$$

Similarly when $f(a)=\pm\infty$.

When the limits in (1.4.22) and (1.4.23) exist, the integrals are said to be *convergent*. If the limits exist with $f(x)$ replaced by $|f(x)|$, then the integrals are *absolutely convergent*. Absolute convergence also implies convergence.

Cauchy Principal Values

Let $c\in(a,b)$ and assume that $\int_a^{c-\epsilon} f(x)\,dx$ and $\int_{c+\epsilon}^b f(x)\,dx$ exist when $0<\epsilon<\min(c-a,b-c)$, but not necessarily when $\epsilon=0$. Then we define

1.4.24
$$\fint_a^b f(x)\,dx = P\int_a^b f(x)\,dx = \lim_{\epsilon\to 0+}\left(\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx\right),$$

when this limit exists.

Similarly, assume that $\int_{-b}^b f(x)\,dx$ exists for all finite values of $b\ (>0)$, but not necessarily when $b=\infty$. Then we define

1.4.25
$$\fint_{-\infty}^\infty f(x)\,dx = P\int_{-\infty}^\infty f(x)\,dx = \lim_{b\to\infty}\int_{-b}^b f(x)\,dx,$$

when this limit exists.

Fundamental Theorem of Calculus

For $F'(x)=f(x)$ with $f(x)$ continuous,

1.4.26 $$\int_a^b f(x)\,dx = F(b)-F(a),$$

1.4.27 $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Change of Variables

If $\phi'(x)$ is continuous or piecewise continuous, then

1.4.28 $$\int_a^b f(\phi(x))\phi'(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(t)\,dt.$$

First Mean Value Theorem

For $f(x)$ continuous and $\phi(x)\geq 0$ and integrable on $[a,b]$, there exists $c\in[a,b]$, such that

1.4.29 $$\int_a^b f(x)\phi(x)\,dx = f(c)\int_a^b \phi(x)\,dx.$$

Second Mean Value Theorem

For $f(x)$ monotonic and $\phi(x)$ integrable on $[a,b]$, there exists $c\in[a,b]$, such that

1.4.30
$$\int_a^b f(x)\phi(x)\,dx = f(a)\int_a^c \phi(x)\,dx + f(b)\int_c^b \phi(x)\,dx.$$

Repeated Integrals

If $f(x)$ is continuous or piecewise continuous on $[a,b]$, then

1.4.31
$$\int_a^b dx_n\int_a^{x_n} dx_{n-1}\cdots\int_a^{x_2} dx_1\int_a^{x_1} f(x)\,dx = \frac{1}{n!}\int_a^b (b-x)^n f(x)\,dx.$$

Square-Integrable Functions

A function $f(x)$ is *square-integrable* if

1.4.32 $$\|f\|_2^2 \equiv \int_a^b |f(x)|^2\,dx < \infty.$$

Functions of Bounded Variation

With $a<b$, the *total variation* of $f(x)$ on a finite or infinite interval (a,b) is

1.4.33 $$\mathcal{V}_{a,b}(f) = \sup\sum_{j=1}^n |f(x_j)-f(x_{j-1})|,$$

where the supremum is over all sets of points $x_0<x_1<\cdots<x_n$ in the *closure* of (a,b), that is, (a,b) with a,b added when they are finite. If $\mathcal{V}_{a,b}(f)<\infty$, then $f(x)$ is of *bounded variation* on (a,b). In this case, $g(x)=\mathcal{V}_{a,x}(f)$ and $h(x)=\mathcal{V}_{a,x}(f)-f(x)$ are nondecreasing bounded functions and $f(x)=g(x)-h(x)$.

If $f(x)$ is continuous on the closure of (a,b) and $f'(x)$ is continuous on (a,b), then

1.4.34 $$\mathcal{V}_{a,b}(f) = \int_a^b |f'(x)\,dx|,$$

whenever this integral exists.

Lastly, whether or not the real numbers a and b satisfy $a<b$, and whether or not they are finite, we *define* $\mathcal{V}_{a,b}(f)$ by (1.4.34) whenever this integral exists. This definition also applies when $f(x)$ is a complex function of the real variable x. For further information on total variation see Olver (1997b, pp. 27–29).

1.4(vi) Taylor's Theorem for Real Variables

If $f(x)\in C^{n+1}[a,b]$, then

1.4.35 $$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n,$$

1.4.36 $$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \qquad a<c<x,$$

and

1.4.37 $$R_n = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt.$$

1.4(vii) Maxima and Minima

If $f(x)$ is twice-differentiable, and if also $f'(x_0) = 0$ and $f''(x_0) < 0$ (> 0), then $x = x_0$ is a local maximum (minimum) (§1.4(iii)) of $f(x)$. The overall maximum (minimum) of $f(x)$ on $[a, b]$ will either be at a local maximum (minimum) or at one of the end points a or b.

1.4(viii) Convex Functions

A function $f(x)$ is *convex* on (a, b) if

1.4.38 $$f((1-t)c + td) \leq (1-t)f(c) + tf(d)$$

for any $c, d \in (a, b)$, and $t \in [0, 1]$. See Figure 1.4.2. A similar definition applies to closed intervals $[a, b]$.

If $f(x)$ is twice differentiable, then $f(x)$ is convex iff $f''(x) \geq 0$ on (a, b). A continuously differentiable function is convex iff the curve does not lie below its tangent at any point.

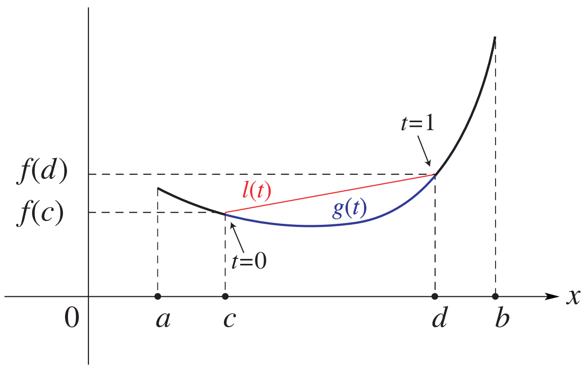

Figure 1.4.2: Convex function $f(x)$. $g(t) = f((1-t)c + td)$, $l(t) = (1-t)f(c) + tf(d)$, $c, d \in (a, b)$, $0 \leq t \leq 1$.

1.5 Calculus of Two or More Variables

1.5(i) Partial Derivatives

A function $f(x, y)$ is *continuous at a point* (a, b) if

1.5.1 $$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b),$$

that is, for every arbitrarily small positive constant ϵ there exists δ (> 0) such that

1.5.2 $$|f(a + \alpha, b + \beta) - f(a, b)| < \epsilon,$$

for all α and β that satisfy $|\alpha|, |\beta| < \delta$.

A function is *continuous on a point set* D if it is continuous at all points of D. A function $f(x, y)$ is *piecewise continuous* on $I_1 \times I_2$, where I_1 and I_2 are intervals, if it is piecewise continuous in x for each $y \in I_2$ and piecewise continuous in y for each $x \in I_1$.

1.5.3 $$\frac{\partial f}{\partial x} = D_x f = f_x = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

1.5.4 $$\frac{\partial f}{\partial y} = D_y f = f_y = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

1.5.5 $$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

The function $f(x, y)$ is *continuously differentiable* if f, $\partial f/\partial x$, and $\partial f/\partial y$ are continuous, *and twice-continuously differentiable* if also $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$, $\partial^2 f/\partial x\, \partial y$, and $\partial^2 f/\partial y\, \partial x$ are continuous. In the latter event

1.5.6 $$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}.$$

Chain Rule

1.5.7 $$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$

1.5.8 $$\frac{\partial}{\partial u} f(x(u, v), y(u, v)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u},$$

1.5.9 $$\frac{\partial}{\partial v} f(x(u, v), y(u, v), z(u, v)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}.$$

Implicit Function Theorem

If $F(x, y)$ is continuously differentiable, $F(a, b) = 0$, and $\partial F/\partial y \neq 0$ at (a, b), then in a *neighborhood* of (a, b), that is, an open disk centered at a, b, the equation $F(x, y) = 0$ defines a continuously differentiable function $y = g(x)$ such that $F(x, g(x)) = 0$, $b = g(a)$, and $g'(x) = -F_x/F_y$.

1.5(ii) Coordinate Systems

Polar Coordinates

With $0 \leq r < \infty$, $0 \leq \phi \leq 2\pi$,

1.5.10 $$x = r\cos\phi, \quad y = r\sin\phi,$$

1.5.11 $$\frac{\partial}{\partial x} = \cos\phi \frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi},$$

1.5.12 $$\frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}.$$

The *Laplacian* is given by

1.5.13 $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2}.$$

Cylindrical Coordinates

With $0 \leq r < \infty$, $0 \leq \phi \leq 2\pi$, $-\infty < z < \infty$,

1.5.14 $$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z.$$

Equations (1.5.11) and (1.5.12) still apply, but

1.5.15 $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

Spherical Coordinates

With $0 \le \rho < \infty$, $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$,

1.5.16 $$x = \rho \sin\theta \cos\phi, \quad y = \rho \sin\theta \sin\phi, \quad z = \rho \cos\theta.$$

The Laplacian is given by

1.5.17 $$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \\ &\quad + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right).\end{aligned}$$

For applications and other coordinate systems see §§12.17, 14.19(i), 14.30(iv), 28.32, 29.18, 30.13, 30.14. See also Morse and Feshbach (1953a, pp. 655-666).

1.5(iii) Taylor's Theorem; Maxima and Minima

If f is $n+1$ times continuously differentiable, then

1.5.18 $$\begin{aligned} f(a+\lambda, b+\mu) &= f + \left(\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial y}\right) f + \cdots \\ &\quad + \frac{1}{n!}\left(\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial y}\right)^n f + R_n,\end{aligned}$$

where f and its partial derivatives on the right-hand side are evaluated at (a,b), and $R_n/(\lambda^2+\mu^2)^{n/2} \to 0$ as $(\lambda,\mu) \to (0,0)$.

$f(x,y)$ has a *local minimum* (*maximum*) at (a,b) if

1.5.19 $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \quad \text{at } (a,b),$$

and the second-order term in (1.5.18) is *positive definite (negative definite)*, that is,

1.5.20 $$\frac{\partial^2 f}{\partial x^2} > 0 \quad (< 0) \quad \text{at } (a,b),$$

and

1.5.21 $$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2 > 0 \quad \text{at } (a,b).$$

1.5(iv) Leibniz's Theorem for Differentiation of Integrals

Finite Integrals

1.5.22
$$\begin{aligned}\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy &= f(x,\beta(x))\beta'(x) - f(x,\alpha(x))\alpha'(x) \\ &\quad + \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}\,dy.\end{aligned}$$

Sufficient conditions for validity are: (a) f and $\partial f/\partial x$ are continuous on a rectangle $a \le x \le b$, $c \le y \le d$; (b) when $x \in [a,b]$ both $\alpha(x)$ and $\beta(x)$ are continuously differentiable and lie in $[c,d]$.

Infinite Integrals

Suppose that a, b, c are finite, d is finite or $+\infty$, and $f(x,y)$, $\partial f/\partial x$ are continuous on the partly-closed rectangle or infinite strip $[a,b] \times [c,d)$. Suppose also that $\int_c^d f(x,y)\,dy$ converges and $\int_c^d (\partial f/\partial x)\,dy$ *converges uniformly* on $a \le x \le b$, that is, given any positive number ϵ, however small, we can find a number $c_0 \in [c,d)$ that is independent of x and is such that

1.5.23 $$\left|\int_{c_1}^{d} (\partial f/\partial x)\,dy\right| < \epsilon,$$

for all $c_1 \in [c_0, d)$ and all $x \in [a,b]$. Then

1.5.24 $$\frac{d}{dx}\int_c^d f(x,y)\,dy = \int_c^d \frac{\partial f}{\partial x}\,dy, \quad a < x < b.$$

1.5(v) Multiple Integrals

Double Integrals

Let $f(x,y)$ be defined on a closed rectangle $R = [a,b] \times [c,d]$. For

1.5.25 $$a = x_0 < x_1 < \cdots < x_n = b,$$

1.5.26 $$c = y_0 < y_1 < \cdots < y_m = d,$$

let (ξ_j, η_k) denote any point in the rectangle $[x_j, x_{j+1}] \times [y_k, y_{k+1}]$, $j = 0, \dots, n-1$, $k = 0, \dots, m-1$. Then the *double integral* of $f(x,y)$ over R is defined by

1.5.27 $$\begin{aligned}&\iint_R f(x,y)\,dA \\ &\quad = \lim \sum_{j,k} f(\xi_j, \eta_k)(x_{j+1} - x_j)(y_{k+1} - y_k)\end{aligned}$$

as $\max((x_{j+1} - x_j) + (y_{k+1} - y_k)) \to 0$. Sufficient conditions for the limit to exist are that $f(x,y)$ is continuous, or piecewise continuous, on R.

For $f(x,y)$ defined on a point set D contained in a rectangle R, let

1.5.28 $$f^*(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D, \\ 0, & \text{if } (x,y) \in R \setminus D.\end{cases}$$

Then

1.5.29 $$\iint_D f(x,y)\,dA = \iint_R f^*(x,y)\,dA,$$

provided the latter integral exists.

If $f(x,y)$ is continuous, and D is the set

1.5.30 $$a \le x \le b, \quad \phi_1(x) \le y \le \phi_2(x),$$

with $\phi_1(x)$ and $\phi_2(x)$ continuous, then

1.5.31 $$\iint_D f(x,y)\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx,$$

where the right-hand side is interpreted as the repeated integral

1.5.32 $$\int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right) dx.$$

Cambridge University Press
978-0-521-14063-8 - NIST Handbook of Mathematical Functions
Edited by Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert and Charles W. Clark
Excerpt
More information

In particular, $\phi_1(x)$ and $\phi_2(x)$ can be constants.

Similarly, if D is the set

1.5.33 $$c \le y \le d, \quad \psi_1(y) \le x \le \psi_2(y),$$

with $\psi_1(y)$ and $\psi_2(y)$ continuous, then

1.5.34 $$\iint_D f(x,y)\,dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy.$$

Change of Order of Integration

If D can be represented in both forms (1.5.30) and (1.5.33), and $f(x,y)$ is continuous on D, then

1.5.35 $$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy.$$

Infinite Double Integrals

Infinite double integrals occur when $f(x,y)$ becomes infinite at points in D or when D is unbounded. In the cases (1.5.30) and (1.5.33) they are defined by taking limits in the repeated integrals (1.5.32) and (1.5.34) in an analogous manner to (1.4.22)–(1.4.23).

Moreover, if a,b,c,d are finite or infinite constants and $f(x,y)$ is piecewise continuous on the set $(a,b) \times (c,d)$, then

1.5.36 $$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy,$$

whenever both repeated integrals exist and at least one is absolutely convergent.

Triple Integrals

Finite and infinite integrals can be defined in a similar way. Often the (x,y,z) sets are of the form

1.5.37 $$a \le x \le b, \quad \phi_1(x) \le y \le \phi_2(x),$$
$$\psi_1(x,y) \le z \le \psi_2(x,y).$$

1.5(vi) Jacobians and Change of Variables

Jacobian

1.5.38 $$\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{vmatrix},$$

1.5.39 $$\frac{\partial(x,y)}{\partial(r,\phi)} = r \quad \text{(polar coordinates)}.$$

1.5.40 $$\frac{\partial(f,g,h)}{\partial(x,y,z)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{vmatrix},$$

1.5.41 $$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2 \sin\theta \quad \text{(spherical coordinates)}.$$

Change of Variables

1.5.42 $$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv,$$

where D is the image of D^* under a mapping $(u,v) \to (x(u,v),y(u,v))$ which is one-to-one except perhaps for a set of points of area zero.

1.5.43 $$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} f(x(u,v,w),y(u,v,w),z(u,v,w)) \times \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

Again the mapping is one-to-one except perhaps for a set of points of volume zero.

1.6 Vectors and Vector-Valued Functions

1.6(i) Vectors

1.6.1 $$\mathbf{a} = (a_1,a_2,a_3), \quad \mathbf{b} = (b_1,b_2,b_3).$$

Dot Product (or Scalar Product)

1.6.2 $$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Magnitude and Angle of Vector a

1.6.3 $$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}},$$

1.6.4 $$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|};$$

θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Unit Vectors

1.6.5 $$\mathbf{i} = (1,0,0), \quad \mathbf{j} = (0,1,0), \quad \mathbf{k} = (0,0,1),$$

1.6.6 $$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Cross Product (or Vector Product)

1.6.7 $$\mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j},$$

1.6.8 $$\mathbf{j}\times\mathbf{i} = -\mathbf{k}, \quad \mathbf{k}\times\mathbf{j} = -\mathbf{i}, \quad \mathbf{i}\times\mathbf{k} = -\mathbf{j}.$$

1.6.9 $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} = \|\mathbf{a}\|\|\mathbf{b}\|(\sin\theta)\mathbf{n},$$

where $\mathbf{n}$ is the unit vector normal to $\mathbf{a}$ and $\mathbf{b}$ whose direction is determined by the right-hand rule; see Figure 1.6.1.

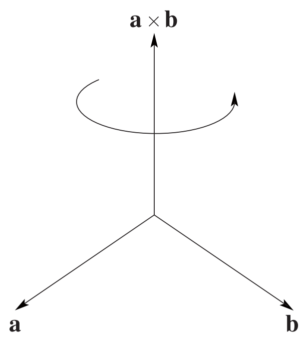

Figure 1.6.1: Vector notation. Right-hand rule for cross products.

Area of parallelogram with vectors $\mathbf{a}$ and $\mathbf{b}$ as sides $= \|\mathbf{a} \times \mathbf{b}\|$.

Volume of a parallelepiped with vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ as edges $= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

1.6.10 $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}),$$

1.6.11 $$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}).$$

1.6(ii) Vectors: Alternative Notations

The following notations are often used in the physics literature; see for example Lorentz *et al.* (1923, pp. 122–123).

Einstein Summation Convention

Much vector algebra involves summation over suffices of products of vector components. In almost all cases of repeated suffices, we can suppress the summation notation entirely, if it is understood that an implicit sum is to be taken over any repeated suffix. Thus pairs of indefinite suffices in an expression are resolved by being summed over (or "traced" over).

Example

1.6.12 $$a_j b_j = \sum_{j=1}^{3} a_j b_j = \mathbf{a} \cdot \mathbf{b}.$$

Next,

1.6.13 $$\boldsymbol{e_1} = (1,0,0), \quad \boldsymbol{e_2} = (0,1,0), \quad \boldsymbol{e_3} = (0,0,1);$$

compare (1.6.5). Thus $a_j \boldsymbol{e_j} = \mathbf{a}$.

Levi-Civita Symbol

1.6.14 $$\epsilon_{jk\ell} = \begin{cases} +1, & \text{if } j,k,\ell \text{ is even permutation of } 1,2,3, \\ -1, & \text{if } j,k,\ell \text{ is odd permutation of } 1,2,3, \\ 0, & \text{otherwise.} \end{cases}$$

Examples

1.6.15 $$\epsilon_{123} = \epsilon_{312} = 1, \quad \epsilon_{213} = \epsilon_{321} = -1, \quad \epsilon_{221} = 0.$$

1.6.16 $$\epsilon_{jk\ell}\epsilon_{\ell mn} = \delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m},$$

where $\delta_{j,k}$ is the Kronecker delta.

1.6.17 $$\boldsymbol{e_j} \times \boldsymbol{e_k} = \epsilon_{jk\ell}\boldsymbol{e_\ell};$$

compare (1.6.8).

1.6.18 $$a_j \boldsymbol{e_j} \times b_k \boldsymbol{e_k} = \epsilon_{jk\ell} a_j b_k \boldsymbol{e_\ell};$$

compare (1.6.7)–(1.6.8).

Lastly, the volume of a parallelepiped with vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ as edges is $|\epsilon_{jk\ell} a_j b_k c_\ell|$.

1.6(iii) Vector-Valued Functions

Del Operator

1.6.19 $$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

The *gradient* of a differentiable scalar function $f(x,y,z)$ is

1.6.20 $$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The *divergence* of a differentiable vector-valued function $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is

1.6.21 $$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

The *curl* of $\mathbf{F}$ is

1.6.22 $$\begin{aligned} \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} \\ &\quad + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \end{aligned}$$

1.6.23 $$\nabla(fg) = f\nabla g + g\nabla f,$$

1.6.24 $$\nabla(f/g) = (g\nabla f - f\nabla g)/g^2,$$

1.6.25 $$\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla f,$$

1.6.26 $$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}),$$

1.6.27 $$\nabla \cdot (\nabla \times \mathbf{F}) = \operatorname{div} \operatorname{curl} \mathbf{F} = 0,$$

1.6.28 $$\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F},$$

1.6.29 $$\nabla \times (\nabla f) = \operatorname{curl} \operatorname{grad} f = 0,$$

1.6.30 $$\nabla^2 f = \nabla \cdot (\nabla f),$$

1.6.31 $$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g),$$

1.6.32 $$\nabla \cdot (\nabla f \times \nabla g) = 0,$$

1.6.33 $$\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f,$$

1.6.34 $$\nabla \times (\nabla \times \mathbf{F}) = \operatorname{curl} \operatorname{curl} \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$