

## **Computational Methods for Multiphase Flow**

Predicting the behavior of multiphase flows is a problem of immense importance for both industrial and natural processes. Thanks to high-speed computers and advanced algorithms, the numerical simulation of such flows is rapidly gaining importance. Researchers and students alike need to have a one-stop account of the field. This book provides a comprehensive and self-contained graduate-level introduction to the computational modeling of multiphase flows. Each chapter is written by a recognized expert and contains extensive references to current research. The book is organized so that the chapters are essentially independent and it can be used for a range of advanced courses and the self-study of specific topics. In the first part a variety of different numerical methods for direct numerical simulations are described and illustrated with suitable examples. The second part is devoted to the numerical treatment of higher-level, averaged-equations models. No other book offers the simultaneous coverage of so many topics related to the computational modeling of multiphase flow. It will be welcomed by researchers and graduate students in engineering, physics, and applied mathematics.

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# Computational Methods for Multiphase Flow

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and  
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Cambridge University Press  
978-0-521-13861-1 — Computational Methods for Multiphase Flow  
Edited by Andrea Prosperetti, Grétar Tryggvason  
Frontmatter  
[More Information](#)

**CAMBRIDGE**  
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom  
One Liberty Plaza, 20th Floor, New York, NY 10006, USA  
477 Williamstown Road, Port Melbourne, VIC 3207, Australia  
314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India  
103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

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[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9780521138611](http://www.cambridge.org/9780521138611)

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First published 2007

Reprinted in paperback with corrections 2009

*A catalogue record for this publication is available from the British Library*

ISBN 978-0-521-84764-3 Hardback

ISBN 978-0-521-13861-1 Paperback

Additional resources for this publication at [www.cambridge.org/comp\\_mult\\_flow](http://www.cambridge.org/comp_mult_flow)

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## Preface

*Computation has made theory more relevant*

This is a graduate-level textbook intended to serve as an introduction to computational approaches which have proven useful for problems arising in the broad area of multiphase flow. Each chapter contains references to the current literature and to recent developments on each specific topic, but the primary purpose of this work is to provide a solid basis on which to build both applications and research. For this reason, while the reader is expected to have had some exposure to graduate-level fluid mechanics and numerical methods, no extensive knowledge of these subjects is assumed. The treatment of each topic starts at a relatively elementary level and is developed so as to enable the reader to understand the current literature.

A large number of topics fall under the generic label of “computational multiphase flow,” ranging from fully resolved simulations based on first principles to approaches employing some sort of coarse-graining and averaged equations. The book is ideally divided into two parts reflecting this distinction. The first part (Chapters 2–5) deals with methods for the solution of the Navier–Stokes equations by finite difference and finite element methods, while the second part (Chapters 9–11) deals with various reduced descriptions, from point-particle models to two-fluid formulations and averaged equations. The two parts are separated by three more specialized chapters on the lattice Boltzmann method (Chapter 6), the boundary integral method for Stokes flow (Chapter 7), and on averaging and the formulation of averaged equation (Chapter 8).

This is a multi-author volume, but we have made an effort to unify the notation and to include cross-referencing among the different chapters. Hopefully this feature avoids the need for a sequential reading of the chapters, possibly aside from some introductory material mostly presented in Chapter 1.

The objective of this work is to describe computational methods, rather than the physics of multiphase flow. With this aspect in mind, the primary criterion in the selection of specific examples has been their usefulness to illustrate the capabilities of an algorithm rather than the characteristics of particular flows. Selected figures are available in color via the book's website at [http://www.cambridge.org/comp\\_mult\\_flow](http://www.cambridge.org/comp_mult_flow).

The original idea for this book was conceived when we chaired the Study Group on Computational Physics in connection with the Workshop on Scientific Issues in Multiphase Flow. The workshop, chaired by Prof. T.J. Hanratty, was sponsored by the U.S. Department of Energy and held on the campus of the University of Illinois at Urbana–Champaign on May 7–9 2002; a summary of the findings has been published in the *International Journal of Multiphase Flow*, Vol. 29, pp. 1041–1116 (2003). As we started to collect material and to receive input from our colleagues, it became clearer and clearer that multiphase flow computation has become an activity with a major impact in industry and research. While efforts in this area go back at least five decades, the great improvement in hardware and software of the last few years has provided a significant impulse which, if anything, can be expected to only gain momentum in the coming years.

Most multiphase flows inherently involve a multiplicity of both temporal and spatial scales. Phenomena at the scale of single bubbles, drops, solid particles, capillary waves, and pores determine the behavior of large chemical reactors, energy production systems, oil extraction, and the global climate itself. Our ability to see how the integration across all these scales comes about and what are its consequences is severely limited by this mind-boggling complexity. This is yet another area where computing offers a powerful tool for significant progress in our ability to understand and predict.

Basic understanding is achieved not only through the simulation of actual physical processes, but also with the aid of computational “experiments.” Multiphase flows are notorious for the difficulties in setting up fully controlled physical experiments. However, computationally, it is possible, for example, to include or not include gravity, account for the effects of a well-characterized surfactant, and others. It is now possible to routinely compute the behavior of relatively simple systems, such as the breakup of jets and the shape of bubbles. The next few years are likely to result in an explosion of results for such relatively simple systems where computations will help us gain a very complete picture of the relevant physics over a large range of parameters. A strong impulse to these activities will be imparted by

effective computational methods for multiscale problems, which are rapidly developing.

At a practical, industrial level, simulation must rely on an averaged description and closure models to account for the unresolved phenomena. The formulation of these closures will greatly benefit from the detailed simulation of the underlying microphysics. The situation is similar to single-phase turbulent flows where, in the last two decades, simulations have played a major role, e.g. in developing large-eddy models.

It is in the examination of very complex, very large-scale systems, where it is necessary to follow the evolution of an enormous range of scales for a long time, that the major challenges and opportunities lie. Such simulations, in which it is possible to get access to the complete data and to control accurately every aspect of the system, will not only revolutionize our predictive capability, but also open up new opportunities for controlling the behavior of such systems.

It is our firm belief that today we stand at the threshold of exciting developments in the understanding of multiphase flows for which computation will prove an essential element. All of us – authors and editors – sincerely hope that this book will contribute to further progress in this field.

Andrea Prosperetti  
Grétar Tryggvason

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## Acknowledgments

The editors and the contributors to the present volume wish to acknowledge the help and support received by several individuals and organizations in connection with the preparation of this work.

- S. BALACHANDAR's research was supported by the ASCI Center for the Simulation of Advanced Rockets at the University of Illinois at Urbana-Champaign through the U.S. Department of Energy (subcontract number B341494).
- JERZY BLAWDZIEWICZ would like to acknowledge the support provided by NSF CAREER grant CTS-0348175.
- HOWARD H. HU's research was supported by NSF grant CTS-9873236 and by DARPA through a grant to the University of Pennsylvania.
- M. YOUSUFF HUSSAINI would like to acknowledge NSF contract DMS 0108672, and the support and encouragement of Provost Lawrence G. Abele.
- LI-SHI LUO would like to acknowledge the support provided by NSF grant CTS-0500213.
- SREEKANTH PANNALA and SANKARAN SUNDARESAN would like to thank Tom O'Brien, Madhava Syamlal and the MFIX team. The contribution has been partly authored by a contractor of the U.S. Government under Contract No. DE-AC05-00OR22725. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes.
- ANDREA PROSPERETTI expresses his gratitude to Drs. Anthony J. Baratta, Cesare Frepoli, Yao-Shin Hwang, Raad Issa, John H. Mahaffy, Randi Moe, Christopher J. Murray, Fadel Moukalled, Sylvain Pigny,

Iztok Tiselj, and Vaughn E. Whisker. His work was supported by NSF grant CTS-0210044 and by DOE grant DE-FG02-99ER14966.

- MARK SUSSMAN's contribution was supported in part by the National Science Foundation under contract DMS 0108672
- GRETAR TRYGGVASON would like to thank his graduate students and collaborators who have contributed to his work on multiphase flows. He would also like to acknowledge support by DOE grant DE-FG02-03ER46083, NSF grant CTS-0522581, as well as NASA projects NAG3-2535 and NNC05GA26G, during the preparation of this book.
- DUAN Z. ZHANG would like to acknowledge many important discussions and physical insights offered by Dr. F. H. Harlow. The Joint DoD/DoE Munitions Technology Development Program provided the financial support for this work.

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