1
Introduction

1.1 Background

Given a group G, we write $\operatorname{Soc} G$ for the *socle* of G: the subgroup of G generated by its minimal normal subgroups. A group G is *almost simple* if $S \leqslant G \leqslant \operatorname{Aut} S$ for some non-abelian simple group S. Note that $S = \operatorname{Soc} G$. A group G is *perfect* if $G = G'$. A group G is *quasisimple* if G is perfect and $G/\mathrm{Z}(G)$ is a non-abelian simple group.

Aschbacher [1] proves a classification theorem, which subdivides the subgroups of the finite classical groups into nine classes. The first eight of these consist roughly of groups that preserve some kind of geometric structure; for example the first class, $\mathscr{C}_1$, consists (roughly) of the reducible groups, which fix a proper non-zero subspace of the vector space on which the group acts naturally. Subgroups of classical groups that lie in the first eight classes are of *geometric type*. The ninth class, denoted by $\mathscr{C}_9$ or $\mathscr{S}$, consists (roughly) of those absolutely irreducible subgroups that are not of geometric type and which, modulo the central subgroup of scalar matrices, are almost simple.

In [66], Kleidman and Liebeck provide an impressively detailed enumeration of the maximal subgroups of geometric type of the finite classical groups of dimension greater than 12. More precisely, they classify the conjugacy classes of maximal subgroups $\bar{H}$ of those almost simple groups $\bar{G}$ for which $\bar{\Omega} := \operatorname{Soc} \bar{G} = \Omega/\mathrm{Z}(\Omega)$ for some classical quasisimple group Ω, with $\bar{H} \cap \bar{\Omega} = K/\mathrm{Z}(\Omega)$ for a subgroup K of Ω of geometric type.

In this book, we determine the maximal subgroups of all such almost simple groups $\bar{G}$ with dimension at most 12. For the subgroups of geometric type, Kleidman and Liebeck proved that their lists contain all such maximal subgroups even in dimensions at most 12. But their determination of when these subgroups are actually maximal applies only to dimensions greater than 12. It turns out that they are nearly all maximal, with just a few exceptions in small dimensions: all of the exceptions are in dimension at most 8.

We do not, however, restrict ourselves to the subgroups of geometric type, and include those subgroups in Aschbacher Class $\mathscr{S}$ in our classification. It is a feature of the groups in this class that they are not, as far as we know, susceptible to a uniform description across all dimensions, but can only be listed for each individual dimension and type of classical group. Fortunately, lists are available of all irreducible representations of degree up to 250 of all finite quasisimple groups G. These have been compiled by Lübeck [84] for representations of G in defining characteristic (when G is a group of Lie type), and by Hiß and Malle [42] for all other representations. These lists provide us with a complete set of candidates for the quasisimple normal subgroups S of maximal subgroups in Class $\mathscr{S}$ of the finite classical groups of dimension up to 250.

We are, however, left with two major problems. Firstly, in order to find the *almost* simple maximal subgroups of the almost simple classical groups $\bar{G} = G/\mathrm{Z}(\Omega)$, we need to determine which of the automorphisms of the simple groups $S/\mathrm{Z}(S)$ in the lists of candidates can be adjoined within $\bar{G}$. Secondly, we need to determine which of the candidates that we construct are actually maximal subgroups of the almost simple groups. Indeed, our approach to the project as a whole follows the same general pattern as [66]: first we find the candidates for the maximal subgroups within each of the nine Aschbacher classes, then we determine which are maximal within their own class, and finally we decide maximality itself by identifying cases in which maximal groups in one class are properly contained in a subgroup in another class.

The $\mathrm{O}_8^+(q)$ case is handled in detail in [62], so we shall not repeat that work here: we will simply reproduce the table of maximal subgroups from [62], but in the format we are using for the remainder of our tables.

Structure of this book. In the remainder of this chapter we present basic results on the structure and representations of simple groups; this material will be required both for the study of geometric type groups and of groups in Class $\mathscr{S}$. Topics covered include: novelty maximal subgroups; finite fields; sesquilinear and quadratic forms, including the specification of our standard forms; introduction to the classical groups, including the specification of our standard outer automorphisms; some relevant representation theory; tensor products; exceptional properties of various small classical groups; permutation and matrix representations of the classical groups; properties of the natural matrix representations of the classical groups; Zsigmondy primes; quadratic reciprocity.

In Chapter 2 we first state our main theorem, Theorem 2.1.1. Then in Section 2.2 we introduce the *types* of geometric subgroups: these are families of subgroups with the property that if H is a geometric maximal subgroup of a quasisisimple classical group, then H is a member of one of these families. For each geometric Aschbacher class, we define the corresponding types, give the structure of the maximal groups of each type, and prove some elementary properties. This detailed definition of the types enables us to state our version of Aschbacher's theorem in Subsection 2.2.9, which is essentially the refined version given in [66]. In Section 2.3 we establish some results concerning maximality of the geometric subgroups that can be proved simultaneously for more than one dimension at a time.

In Chapter 3 we consider each dimension from 2 to 12 in turn, and determine which subgroups are maximal amongst the geometric subgroups of the almost simple classical groups of that dimension. Thus by the end of this chapter we have produced a list of candidate geometric maximals, which only need to be compared with the Class $\mathscr{S}$ groups to determine their maximality.

Section 4.1 presents our overall strategy for classifying the maximal subgroups in Class $\mathscr{S}$, and a description of the subdivision between Class $\mathscr{S}_1$ (cross characteristic) and Class $\mathscr{S}_2$ (defining characteristic). The remainder of Chapter 4 is devoted to finding the maximal subgroups in Class $\mathscr{S}_1$. There is a section (4.2) on working with algebraic irrationalities. After this we present in Section 4.3 our list of $\mathscr{S}_1$-candidates (taken from [42]). The next few sections concern dimensions 2 to 6, and alternate between theory and practice. In Sections 4.4 and 4.5, we first describe how to calculate the stabiliser of a quasisimple group in cross characteristic in a conformal classical group (the group of all linear mappings that multiply the form by a scalar), and hence to determine the exact structure and conjugacy behaviour of a candidate maximal $\mathscr{S}_1$-subgroup of a quasisimple group, and then carry out these calculations in some detail in dimensions up to 6. Next, in Sections 4.6 and 4.7 we present methods to determine the actions of the duality and field automorphisms of a quasisimple classical group on its quasisimple subgroups, and apply these methods in dimensions up to 6. In Section 4.8 we then determine whether any of these $\mathscr{S}_1$-subgroups contain one another, and hence are non-maximal. After carrying out all of these calculations in detail in dimensions up to 6, in Section 4.9 we perform the same calculations for dimensions 7 to 12. This means that we have determined those subgroups of the almost simple classical groups in dimension at most 12 that are maximal amongst the $\mathscr{S}_1$-subgroups: this list is summarised in Section 4.10.

In Chapter 5 we move on to considering the maximal subgroups arising from representations of groups of Lie type in defining characteristic. Note that we in fact define a class $\mathscr{S}_2^*$ of subgroups (see Definition 5.1.15) and work only with them. In Section 5.1 we present as much of the general theory of representations of groups of Lie type in defining characteristic as we will need to perform our calculations. In Section 5.2 we briefly present some information about symmetric and anti-symmetric powers of modules. We then consider the possible families of groups of Lie type in turn, basing our lists of candidates on

[84]. For each possible candidate maximal subgroup we determine the structure of the maximal such subgroup of the quasisimple group, the module on which it acts, the number of conjugacy classes in the quasisimple group, and the stabiliser of one such class in the conformal classical group. Then in Section 5.3 we consider groups with non-abelian composition factor $\mathrm{L}_2(q)$, in Section 5.4 the groups $\mathrm{L}_n(q)$ and $\mathrm{U}_n(q)$ with $n \geqslant 3$, in Section 5.5 the groups $\mathrm{S}_n(q)$, in Section 5.6 the groups $\mathrm{O}_n^{\varepsilon}(q)$ and ${}^3\mathrm{D}_4(q)$, and finally in Section 5.7 the remaining groups of Lie type. We summarise our findings to this point in Section 5.8. Next, in Section 5.9 we determine the action of duality and field automorphisms on $\mathscr{S}_2^*$-subgroups. In Section 5.10 we determine all containments between the $\mathscr{S}_2^*$-subgroups, and then finally in Section 5.11 we summarise the results of this chapter. Thus Chapter 5 determines all subgroups that are maximal amongst the $\mathscr{S}_2^*$-subgroups in dimension at most 12.

By the end of Chapter 5 we have produced a list of subgroups of the almost simple classical groups such that all maximal subgroups lie in this list. We then proceed in Chapter 6 to determine the containments between these subgroups, and hence to prove the main theorem of the book, Theorem 2.1.1. In Section 6.2 we determine all containments between $\mathscr{S}_1$-maximals and $\mathscr{S}_2^*$-maximals, to produce a set of $\mathscr{S}^*$-maximals (where $\mathscr{S}^* = \mathscr{S}_1 \cup \mathscr{S}_2^*$), and then in Section 6.3 we determine all containments between geometric and $\mathscr{S}^*$-maximals.

Aschbacher's theorem does not apply to certain extensions of $\mathrm{S}_4(2^i)$ and $\mathrm{O}_8^+(q)$ (that is, those that involve the exceptional graph automorphism and the triality graph automorphism, respectively): note, however, that Aschbacher's paper [1] includes a variant of his theorem for the relevant extensions of $\mathrm{S}_4(2^i)$, but we shall not deem this variant to be part of "Aschbacher's theorem". Since the $\mathrm{O}_8^+(q)$ case is fully handled in [62], we do not concern ourselves with that. In Chapter 7 we calculate the maximal subgroups of those almost simple extensions of $\mathrm{S}_4(2^i)$ to which Aschbacher's theorem does not apply, as well as the maximal subgroups of the finite almost simple exceptional groups that have a faithful projective representation in defining characteristic of degree at most 12, namely those with socles ${}^2\mathrm{B}_2(q) = \mathrm{Sz}(q)$, $\mathrm{G}_2(q)$, ${}^2\mathrm{G}_2(q) = \mathrm{R}(q)$ and ${}^3\mathrm{D}_4(q)$. For many of these groups such classifications are already known, and we merely provide references to the original calculations, however we occasionally include our own proofs if we feel this may be helpful for the reader. Finally, in Chapter 8, we present tables of our results.

1.2 Notation

Here we list some general notation used in the book. More specialised notation will be introduced as it arises, and in particular our notation for the classical groups is presented in Subsection 1.6.3 and for the outer automorphisms of the classical groups in Subsection 1.7.1.

By $[a, b]$ we denote the least common multiple of two positive integers a and b, and by (a, b) we denote their greatest common divisor. If $a \in \mathbb{N}$ and p is a prime, we write $(a)_p$ for the highest power of p that divides a. In Section 1.13 we define the notion of a *Zsigmondy prime* for $q^n - 1$, where q is a prime power and $n \geqslant 3$ is an integer. We shall denote such a prime by $z_{q,n}$.

If we write δ_{ij} (with two subscripts), we will mean the Kronecker delta: $\delta_{ii} = 1$ for all i and $\delta_{ij} = 0$ whenever $i \neq j$.

For group elements g and h, we write g^h for $h^{-1}gh$ and $[g, h]$ for $g^{-1}h^{-1}gh$. As usual, $\mathrm{Z}(G)$ is the centre of the group G, and $\mathrm{N}_G(H)$ and $\mathrm{C}_G(H)$ are the normaliser and centraliser of H in G. The derived subgroup of G is written as $[G, G]$ or G', and we define $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$ for $n > 1$, and use G^∞ to denote $\bigcap_{i \geqslant 0} G^{(i)}$. We write $\mathrm{Aut}\, G$ for the automorphism group of G, and $\mathrm{Inn}\, G$ and $\mathrm{Out}\, G$ respectively for its inner and outer automorphism group.

Our notation for group structures is based on that in the ATLAS [12]. Note in particular that this means that we generally use ATLAS notation for simple groups. Thus, for example, $A \times B$ is the direct product of groups A and B, we write $A{:}B$ to denote a split extension of A by B, we write $A{}^{\cdot}B$ to denote a non-split extension (or possibly one in which A is trivial), and we write $A.B$ when we do not know or do not wish to specify whether the extension splits. The symbol $A \wr B$ is defined for an arbitrary group A and a permutation group B, and denotes the wreath product of A by B. If G is a group with a unique index 2 subgroup H, then we sometimes write $\frac{1}{2}G$ to denote H.

The cyclic group of order n is denoted by C_n or (particularly when as a component of a group structure) just by n. An elementary abelian group of order p^n is denoted by E_{p^n} or just by p^n. By $[n]$ we denote a group of order n, of unspecified structure. For elementary abelian groups A we write A^{m+n} to mean a group with an elementary abelian normal subgroup A^m such that the quotient is isomorphic to A^n. The group A^{m+n} is usually, but not always, special. For n even, D_n denotes the dihedral group of order n, and for n a power of 2, we write Q_n for the quaternion group of order n. For r an odd prime, we write r_+^{1+2n} for an extraspecial group of order r^{1+2n} and exponent r, and r_-^{1+2n} for an extraspecial group of the same order, but exponent r^2. We write 2_+^{1+2m} for an extraspecial group of order 2^{1+2m} that is isomorphic to a central product of m copies of D_8, and we write 2_-^{1+2m} for an extraspecial group of the same order, but that is isomorphic to a central product of $m-1$ copies of D_8 and one of Q_8.

For L an arbitrary finite group, $P(L)$ denotes the minimum degree of a non-trivial permutation representation of L.

We write $\mathbb{F}_q$ for a finite field of order $q = p^e$, with a fixed primitive element

$\omega = \omega_q$, and Frobenius automorphism $\phi : x \mapsto x^p$. For a field F, we write $F^\times$ to denote the multiplicative group of F, $\operatorname{char} F$ to denote the characteristic of F, and $(F, +)$ to denote the additive group of F.

As usual, I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ matrix with all entries 1. We write $\operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ for the matrix $A = (a_{ij})_{n\times n}$ with $a_{ii} = \alpha_i$ for all i and $a_{ij} = 0$ otherwise. We write $\operatorname{antidiag}(\alpha_1, \ldots, \alpha_n)$ for the matrix $A = (a_{ij})_{n\times n}$ with $a_{i,n-i+1} = \alpha_i$ for all i and $a_{ij} = 0$ otherwise. The transpose of A is denoted by A^T, and its trace by $\operatorname{tr}(A)$. We denote the elementary matrix with a 1 in position (i, j) and 0 in all other positions by $\mathrm{E}_{i,j}$.

The set of all $(m \times n)$-matrices with entries in the field F is denoted by $\mathrm{M}_{m\times n}(F)$, or by $\mathrm{M}_{m\times n}(q)$ when $F = \mathbb{F}_q$. For a vector space V over a field F, we write $\mathrm{GL}(V)$ for the *general linear group* of V, which is the set of all invertible F-linear maps from V to itself. For a given basis of V, we can identify $\mathrm{GL}(V)$ with $\mathrm{GL}_n(F)$ (or $\mathrm{GL}_n(q)$ when $F = \mathbb{F}_q$), the set of all invertible $n \times n$ matrices over F. Our convention is that linear maps act on the right, with corresponding action of matrices on row vectors by right multiplication.

We write $\mathrm{GL}^\pm_n(q)$, and related notation such as $\mathrm{SL}^\pm_n(q)$, to denote the linear and unitary groups: the $+$ sign corresponds to the linear groups, and the $-$ sign to the unitary groups. (We may also of course denote the unitary groups by the more usual $\mathrm{GU}_n(q)$ and $\mathrm{SU}_n(q)$.)

If U and W are subspaces of V, and $G \leqslant \mathrm{GL}(V)$, then we write $\mathrm{N}_G(W)$ for the stabiliser in G of W, and $\mathrm{N}_G(W, U)$ for $\mathrm{N}_G(W) \cap \mathrm{N}_G(U)$. If $G \leqslant S_n$ stabilises a set $W \subseteq \{1, \ldots, n\}$, or $G \leqslant \mathrm{GL}(V)$ stabilises a subspace $W \leqslant V$, then by G^W we mean the image of the induced action of G on W.

For a vector space V, we write V^* for the dual space of V. If V is equipped with a reflexive form β (see Section 1.5), then we write $V = A \perp B$ to mean that V decomposes as a direct sum of A and B, such that $\beta(a, b) = 0$ for all $a \in A$, $b \in B$. For $v \in V$, we write $v^\perp$ for the subspace $\{\, w \in V \mid \beta(v, w) = 0 \,\}$, and similarly $W^\perp = \{\, v \in V \mid \beta(w, v) = 0 \text{ for all } w \in W \,\}$. If Q is a quadratic form on V, and W is a non-degenerate subspace of V, then $\operatorname{sgn}(W)$ denotes the sign of the restriction of Q to W (so $\operatorname{sgn}(W)$ can be $\circ$, $+$ or $-$).

1.3 Some basic group theory

An *automorphism* of a group G is a bijective homomorphism from G to itself: the set of all automorphisms of G forms a group, $\operatorname{Aut} G$. For a fixed $g \in G$, we denote the automorphism $x \mapsto x^g = g^{-1}xg$ by c_g. An automorphism ϕ of G is *inner* if there exists a g in G such that $\phi = c_g$. We denote the group of all inner automorphisms of G by $\operatorname{Inn} G$. Note that $\operatorname{Inn} G \cong G/\mathrm{Z}(G)$. It is an easy exercise to prove that $\operatorname{Inn} G \trianglelefteq \operatorname{Aut} G$; the quotient $\operatorname{Aut} G/\operatorname{Inn} G$ is

$\mathrm{Out}\, G$, the *outer automorphism group* of G. An *outer automorphism* of G is often defined to be an element of $\mathrm{Aut}\, G \setminus \mathrm{Inn}\, G$ rather than an element of $\mathrm{Out}\, G$. In this book, however, despite the risk of causing confusion, we find it convenient to use "outer automorphism" to denote either a non-trivial element α of $\mathrm{Out}\, G$ (which is consequently only defined modulo $\mathrm{Inn}\, G$, and hence not itself an automorphism of G), or a representative α of a non-trivial coset of $\mathrm{Inn}\, G$ in $\mathrm{Aut}\, G$, depending on the context.

Let $\alpha \in \mathrm{Aut}\, G$, let C be a conjugacy class of G, and let $C^\alpha = \{x^\alpha : x \in C\}$. We say that α *stabilises* C if $C^\alpha = C$. The following lemma is elementary.

Lemma 1.3.1 *Let $\alpha \in \mathrm{Aut}\, G$ and let C be a conjugacy class of G. Then C^α is also a conjugacy class of G. Furthermore, if α stabilises C and $x \in C$, then there exists $g \in G$ with $x^{\alpha c_g} = x$.*

So, for a given C, the class C^α depends only on the coset of $\mathrm{Inn}\, G$ in which α lies, and hence there is an induced action of $\mathrm{Out}\, G$ on the set of conjugacy classes of G. So when we write C^α with α an outer automorphism of G or talk about an outer automorphism stabilising C, then it does not matter whether we are thinking of α as an element of $\mathrm{Aut}\, G$ or of $\mathrm{Out}\, G$.

The following theorem is a straightforward consequence of the classification of finite simple groups, and was known as the *Schreier Conjecture* before the completion of the classification.

Theorem 1.3.2 *Let S be a finite non-abelian simple group. Then $\mathrm{Out}\, S$ is soluble.*

We will occasionally need the concept of isoclinism. The commutator $[x, y]$ of two elements x and y of a group G is unchanged if we multiply x and y by central elements of G. Thus we can think of the commutator map not as a map from $G \times G$ to G, but instead as a map from $G/\mathrm{Z}(G) \times G/\mathrm{Z}(G)$ to G.

Definition 1.3.3 Two groups G and H are *isoclinic* if there are isomorphisms $\rho : G/\mathrm{Z}(G) \to H/\mathrm{Z}(H)$ and $\theta : G' \to H'$ which form a commutative diagram with the commutator maps from $G/\mathrm{Z}(G) \times G/\mathrm{Z}(G)$ to G' and from $H/\mathrm{Z}(H) \times H/\mathrm{Z}(H)$ to H'.

$$\begin{array}{ccc} \frac{G}{Z(G)} \times \frac{G}{Z(G)} & \xrightarrow{(\rho,\rho)} & \frac{H}{Z(H)} \times \frac{H}{Z(H)} \\ \Big\downarrow [\,,\,] & & \Big\downarrow [\,,\,] \\ G' & \xrightarrow{\theta} & H' \end{array}$$

The dihedral group of order 8 and the quaternion group of order 8 are isoclinic; see just before Subsection 1.3.1 for another example.

If G is finite, then $G^\infty = \cap_{i=1}^{\infty} G^{(i)}$ is the first perfect group in the derived series of G. If S is non-abelian and simple, and $S \trianglelefteq G \leqslant \mathrm{Aut}\, S$, then Theorem 1.3.2 implies that $G^\infty = S$.

Recall from Section 1.1 that a group G is quasisimple if G is perfect with $G/\mathrm{Z}(G)$ a non-abelian simple group. We shall use the following lemma implicitly throughout much of the book, without further citation.

Lemma 1.3.4 *Let $G = Z^{\cdot}S$ be quasisimple, with Z central and S non-abelian simple. Then* $\mathrm{Aut}\, G$ *can be naturally regarded as a subgroup of* $\mathrm{Aut}\, S$.

Proof Let α be a non-trivial element of $\mathrm{Aut}\, G$. If α induces the identity map on G/Z, then α acts as the identity on all commutators $[g, h]$, and so induces the identity on all of G. Thus $\mathrm{Aut}\, G$ acts faithfully on G/Z. □

A *stem extension* of a group G by a group K is a group $C = K^{\cdot}G$ such that $K \leqslant \mathrm{Z}(C) \cap C'$. In particular, a quasisimple group is a stem extension of a non-abelian simple group.

Definition 1.3.5 The *Schur multiplier* $M(G)$ of a group G is the largest K such that there exists a stem extension of G by K.

It is not immediately apparent that $M(G)$ is well-defined, but it turns out that the corresponding stem extension is determined up to isoclinism by G, (see Definition 1.3.3 of isoclinism), and up to isomorphism if G is perfect.

As an example, the symmetric groups S_n for $n \geqslant 4$ have two isoclinic double covers, $2^{\cdot}\mathrm{A}_n.2^+$ and $2^{\cdot}\mathrm{A}_n.2^-$, whilst by the above remark for $n \geqslant 5$ and $n \neq 6$ the groups A_n have a unique double cover (which is true also for A_4.) This notation comes from [12], and $2^{\cdot}\mathrm{A}_5.2^+$ is the group of which the character table is printed there. The inverse images of transpositions in S_5 have orders 2 and 4 in $2^{\cdot}\mathrm{A}_5.2^+$ and $2^{\cdot}\mathrm{A}_5.2^-$, respectively.

1.3.1 Maximal subgroups of almost simple groups

Let G be an almost simple group with socle S, and let M be a maximal subgroup of G. The group M is a *triviality* if $S \trianglelefteq M$. The trivialities correspond to the maximal subgroups of the soluble group G/S and are very easy to determine. They are consequently generally omitted from tables of maximal subgroups of almost simple groups (for example, in the Atlas [12]), and are excluded from the statements of Aschbacher's theorem and the O'Nan–Scott Theorem.

The following result is fundamental in the analysis of maximal subgroups of almost simple groups. It was first proved by Wilson [113], though we follow the proof of Liebeck, Praeger and Saxl [79, end of paper].

Theorem 1.3.6 *Let G be a finite almost simple group with socle S. Suppose that M is a maximal subgroup of G. Then $S \cap M \neq 1$.*

Proof If $G = S$ then the result is trivial, so suppose not, and assume that $S \cap M = 1$. Then $M < \langle S, M\rangle \leqslant G$, whence $G = SM = S{:}M$, by the maximality of M. Let N be a minimal normal subgroup of M. Then N is characteristically simple. One of the following cases must arise:

(i) $N \cong \mathrm{E}_{p^r}$, where p is prime, $r \geqslant 1$ and $p \mid |S|$;
(ii) $N \cong \mathrm{E}_{p^r}$, where p is prime, $r \geqslant 1$ and $p \nmid |S|$;
(iii) $N \cong T \times \cdots \times T \cong T^m$, where $m \geqslant 1$ and T is non-abelian simple.

The quotient G/S is soluble by Theorem 1.3.2, so Case (iii) does not arise.

In all cases, $M \leqslant \mathrm{N}_G(N)$. Moreover $\mathrm{N}_G(N) \cap S = \mathrm{C}_S(N)$, and of course $\mathrm{C}_S(N) \neq S$, since N is a subgroup of $\mathrm{Aut}\, S$. Thus $\mathrm{N}_G(N) \neq G$ and hence the maximality of M implies that $\mathrm{N}_G(N) = M$ and so $\mathrm{C}_S(N) = 1$.

In Case (i), the conjugation action of N on S centralises 1_S, so must centralise some non-trivial elements of S (since the orbits of N have p-power order and $p \mid |S|$). This contradicts $\mathrm{C}_S(N) = 1$.

In Case (ii), we let $q \mid |S|$, with q prime (so $q \neq p$), and we let Q be a Sylow q-subgroup of S normalised by N, which exists as the number of Sylow q-subgroups of S is a divisor of $|S|$, and therefore not divisible by p. Suppose that N also normalises a Sylow q-subgroup Q_1 of S. Then $Q_1 = Q^{x^{-1}}$ for some $x \in S$. So now N and N^x normalise Q and so are Sylow p-subgroups of $\mathrm{N}_{SN}(Q) = \mathrm{N}_S(Q)N$. Hence there exist $y \in \mathrm{N}_S(Q)$ and $z \in N$ such that $N^{xyz} = N = N^z$, and so there exists $y \in \mathrm{N}_S(Q)$ such that $N^{xy} = N$. Now $[g, xy] \in N \cap S = 1$ for all $g \in N$, and so $xy \in \mathrm{C}_S(N) = 1$. Hence $x = y^{-1} \in \mathrm{N}_S(Q)$, and so $Q_1 = Q$. Therefore $M = \mathrm{N}_G(N) \leqslant \mathrm{N}_G(Q)$, because N normalises a unique Sylow q-subgroup of S, and so $M < QM < SM = G$, contradicting the maximality of M. □

In Chapter 7, we shall need the following immediate corollary.

Corollary 1.3.7 *Let G be a finite almost simple group with socle S, and let M be a maximal subgroup of G such that $S \not\leqslant M$. Then there exists a characteristically simple group N with $1 < N < S$ such that $M = \mathrm{N}_G(N)$.*

Proof By the previous result, we may assume that $M \cap S \neq 1$. Notice that $M \cap S \trianglelefteqslant M$, so we may choose N to be minimal subject to $1 < N \leqslant M \cap S$ and $N \trianglelefteqslant M$. Thus N is a minimal normal subgroup of M, and so is characteristically simple. Clearly $M \leqslant \mathrm{N}_G(N)$, and from $N \leqslant M \cap S < S$ we deduce that $S \not\leqslant \mathrm{N}_G(N)$. Maximality of M then gives $M = \mathrm{N}_G(N)$. □

Definition 1.3.8 A maximal subgroup M of an almost simple group G is called an *ordinary* maximal subgroup if $S \cap M$ is a maximal subgroup of S. We say that M is a *novel* maximal subgroup (or, simply, *a novelty*) if $S \cap M$ is non-maximal in S.

Suppose that $H < S$ and we are considering $M = \mathrm{N}_G(H)$ as a candidate for being a novel maximal subgroup of G. This is only possible if $\mathrm{N}_G(H)S = G$, so we shall assume that to be the case. Then M fails to be maximal in G if and only if $M < \mathrm{N}_G(K)$ for some K such that $H < K < S$. By replacing H by $\mathrm{N}_S(H)$, we may assume that $\mathrm{N}_S(H) = H$, and similarly we may restrict attention to groups K with $\mathrm{N}_S(K) = K$.

If $\mathrm{N}_G(K)S \neq G$ for some such K then M is not a proper subgroup of $\mathrm{N}_G(K)$. This motivates the following definition.

Definition 1.3.9 Let G be almost simple with socle S. If $H = \mathrm{N}_S(H) < K = \mathrm{N}_S(K) < S < G$, but $\mathrm{N}_G(K)S \neq G$, then $M = \mathrm{N}_G(H)$ is called a *type 1 novelty* with respect to K.

An example is $G = \mathrm{PGL}_2(7)$, $S = \mathrm{L}_2(7)$, $H = \mathrm{D}_6$, $M = \mathrm{D}_{12}$. The only possibility for K is S_4, but $\mathrm{N}_G(K) = K$ in that case.

The following is essentially equivalent to [113, Proposition 2.3 (e),(f)], but in a slightly different context.

Proposition 1.3.10 *Let G be almost simple with socle S. Suppose that G has subgroups $H < K < S < G$, with $\mathrm{N}_S(H) = H$, $\mathrm{N}_S(K) = K$, and $\mathrm{N}_G(H)S = \mathrm{N}_G(K)S = G$. Then $\mathrm{N}_G(H) \not\leq \mathrm{N}_G(K)$ if and only if there exists $H_0 < K$ with H and H_0 conjugate in $\mathrm{N}_G(K)$ but not in K. In this situation H and H_0 are also conjugate in S.*

Proof Let M denote $\mathrm{N}_G(H)$. Suppose first that $M \not\leq \mathrm{N}_G(K)$, and let m be an element of $M \setminus \mathrm{N}_G(K)$. Then $H = H^m < K^m \neq K$, and $\mathrm{N}_G(K)S = G$ implies that $m = ns$ for some $n \in \mathrm{N}_G(K)$ and $s \in S$, and hence that $K^m = K^s$. So $H_0 := H^{s^{-1}} < K$, and $H^n = H^{ms^{-1}} = H_0$, so H and H_0 are conjugate in $\mathrm{N}_G(K)$ and in S. But if $H_0^k = H$ with $k \in K$, then $s^{-1}k \in \mathrm{N}_S(H) = H$, so $s \in K$ and hence $K^m = K^s = K$, a contradiction.

Suppose conversely that $H_0 < K$, where H_0 is such that H and H_0 are conjugate in $\mathrm{N}_G(K)$ but not in K, and let $n \in \mathrm{N}_G(K)$ with $H^n = H_0$. Since $MS = G$, we can write $n = ms$ with $m \in M$ and $s \in S$. If $m \in \mathrm{N}_G(K)$, then $s \in \mathrm{N}_S(K) = K$, so $H_0 = H^n = H^{ms} = H^s$, contradicting the assumption that H and H_0 are not K-conjugate. So $m \notin \mathrm{N}_G(K)$ and hence $M \not\leq \mathrm{N}_G(K)$. □

Definition 1.3.11 Let G be almost simple with socle S. If $H < K < S < G$, with $\mathrm{N}_S(H) = H$, $\mathrm{N}_S(K) = K$, $\mathrm{N}_G(H)S = \mathrm{N}_G(K)S = G$, and such that $M = \mathrm{N}_G(H) \not\leq \mathrm{N}_G(K)$, then M is a *type 2 novelty* (or *Wilson novelty*) with respect to K.