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London Mathematical Society Lecture Note series: 381

Symmetries and Integrability of Difference Equations

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

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www.cambridge.org
Information on this title: www.cambridge.org/9780521136587

First published 2011

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging in Publication data
Symmetries and integrability of difference equations / edited by Decio Levi . . . [et al.].
p. cm. – (London Mathematical Society lecture note series ; 381)
Includes bibliographical references.
ISBN 978-0-521-13658-7 (pbk.)
1. Difference equations. 2. Symmetry (Mathematics) 3. Integrals. I. Levi, D. (Decio)
QA431.S952 2011
515´.625 – dc22 2011006852

ISBN 978-0-521-13658-7 Paperback

Contents

Cambridge University Press & Assessment
978-0-521-13658-7 — Symmetries and Integrability of Difference Equations
Edited by Decio Levi , Peter Olver , Zora Thomova , Pavel Winternitz
Frontmatter
More Information

List of figures

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Preface

This book is based upon lectures delivered during the Summer School on Symmetries and Integrability of Difference Equations at the Université de Montréal, Canada, June 8, 2008–June 21, 2008. The lectures are devoted to methods that have been developed over the last 15–20 years for discrete equations. They are based on either the inverse spectral approach or on the application of geometric and group theoretical techniques. The topics covered in this volume can be summarized in the following categories:

- Integrability of difference equations
- Discrete differential geometry
- Special functions and their relation to continuous and discrete Painlevé functions
- Discretization of complex analysis
- General aspects of Lie group theory relevant for the study of difference equations. Specifically, two such subjects are treated: 1. Cartan's method of moving frames 2. Lattices in Euclidean space, symmetrical under the action of semisimple Lie groups
- Lie point symmetries and generalized symmetries of discrete equations

Twelve distinct lecture series were presented at the Summer School of which eleven are included in this volume. Close to 50 registered graduate students and researchers from twelve different countries participated.

The Summer School, Séminaire de mathématiques supérieures, is a yearly event at the Département de Mathématiques, Université de Montréal. The organizing committee for the year 2008 consisted of Pavel Winternitz (Université de Montréal, Canada), Vladimir Dorodnitsyn (Keldysh Institute of Applied Mathematics, Russian Academy of Sciences), Decio Levi (Universitá degli Studi Roma Tre, Italy) and Peter Olver (University of Minnesota, USA). The two scientific directors were Pavel Winternitz and Vladimir Dorodnitsyn. The

financial support for the Summer School was provided by Université de Montréal, the National Foundation Grant No. 0737765, the Centre de Recherches Mathématiques, and the Institut de Sciences Mathématiques.

The editors would like to thank André Montpetit (CRM) for technical help in preparing this volume. We also acknowledge the support of David Tranah, Suzanne Parker and Clare Dennison from Cambridge University Press.