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Symmetries and Integrability of Difference Equations

Edited by

DECIO LEVI

Università degli Studi Roma Tre

PETER OLVER

University of Minnesota

ZORA THOMOVA

SUNY Institute of Technology

PAVEL WINTERNITZ

Université de Montréal



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Contributors

- Vladimir Dorodnitsyn *Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya Pl. 4, Moscow, 125047, Russia; dorod@spp.keldysh.ru*
- Basile Grammaticos *IMNC, Université Paris VII & XI, CNRS, UMR 8165, Bât. 104, 91406 Orsay, France; grammati@paris7.jussieu.fr*
- Jarmo Hietarinta *Department of Physics and Astronomy, University of Turku, 20014 Turku, Finland; hietarin@utu.fi*
- Mourad E. H. Ismail *Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA and Department of Mathematics, King Saud University, Riyadh, Saudi Arabia; ismail@math.ucf.edu*
- Alexander Its *Department of Mathematical Sciences, 402 N. Blackford Street, Indiana University–Purdue University Indianapolis, Indianapolis, IN 46202-3216, USA; itsa@math.iupui.edu*
- Roman Kozlov *Department of Finance and Management Science, Norwegian School of Economics and Business Administration, Helleveien 30, N-5045, Bergen, Norway; Roman.Kozlov@nhh.no*
- Decio Levi *Dipartimento di Ingegneria Elettronica, Università degli Studi Roma Tre and Sezione INFN Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy; levi@roma3.infn.it*
- Sergey P. Novikov *University of Maryland, College Park, USA, Landau Institute and Steklov Institute, Moscow, Russia; novikov@ipst.umd.edu*
- Peter J. Olver *School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA; olver@math.umn.edu*
- Jiří Patera *Département de mathématiques et statistique and Centre de recherches mathématiques, Université de Montréal, C.P. 6128, succ. Centre ville, Montréal, H3C 3J7, QC Canada; patera@crm.umontreal.ca*
- Alfred Ramani *Centre de Physique Théorique, École Polytechnique, CNRS, 91128 Palaiseau, France; ramani@cph.t.polytechnique.fr*

Contributors

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Yuri B. Suris *Institut für Mathematik, MA 7-2, Technische Universität Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany; suris@math.tu-berlin.de*

Pavel Winternitz *Département de mathématiques et statistique and Centre de recherches mathématiques, Université de Montréal, C.P. 6128, succ. Centre-ville, Montréal, QC H3C 3J7 Canada; wintern@crm.umontreal.ca*

Ravil I. Yamilov *Ufa Institute of Mathematics, Russian Academy of Sciences, 112 Chernyshevsky Street, Ufa 450077, Russian Federation; Rvlyamilov@matem.anrb.ru*

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Preface

This book is based upon lectures delivered during the Summer School on Symmetries and Integrability of Difference Equations at the Université de Montréal, Canada, June 8, 2008–June 21, 2008. The lectures are devoted to methods that have been developed over the last 15–20 years for discrete equations. They are based on either the inverse spectral approach or on the application of geometric and group theoretical techniques. The topics covered in this volume can be summarized in the following categories:

- Integrability of difference equations
- Discrete differential geometry
- Special functions and their relation to continuous and discrete Painlevé functions
- Discretization of complex analysis
- General aspects of Lie group theory relevant for the study of difference equations. Specifically, two such subjects are treated: 1. Cartan's method of moving frames 2. Lattices in Euclidean space, symmetrical under the action of semisimple Lie groups
- Lie point symmetries and generalized symmetries of discrete equations

Twelve distinct lecture series were presented at the Summer School of which eleven are included in this volume. Close to 50 registered graduate students and researchers from twelve different countries participated.

The Summer School, Séminaire de mathématiques supérieures, is a yearly event at the Département de Mathématiques, Université de Montréal. The organizing committee for the year 2008 consisted of Pavel Winternitz (Université de Montréal, Canada), Vladimir Dorodnitsyn (Keldysh Institute of Applied Mathematics, Russian Academy of Sciences), Decio Levi (Università degli Studi Roma Tre, Italy) and Peter Olver (University of Minnesota, USA). The two scientific directors were Pavel Winternitz and Vladimir Dorodnitsyn. The

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