1

Lagrangian and Hamiltonian Formalism for Discrete Equations: Symmetries and First Integrals

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Abstract

In this chapter the relation between symmetries and first integrals of discrete Euler–Lagrange and discrete Hamiltonian equations is considered. These results are built on those for continuous Euler–Lagrange and canonical Hamiltonian equations. First, the well-known Noether theorem which provides conservation laws for continuous Euler–Lagrange equations is reviewed. Then, its discrete analog is presented. Further, it is mentioned that continuous and discrete Hamiltonian equations can be obtained by the variational principle from action functionals. This is used to develop Noether-type theorems for canonical Hamiltonian equations and their discrete counterparts (discrete Hamiltonian equations). The approach based on symmetries of the discrete action functionals provides a simple and clear way to construct first integrals of discrete Euler–Lagrange and discrete Hamiltonian equations by means of differentiation of discrete Lagrangian (or Hamiltonian) and algebraic manipulations. It can be used to conserve structural properties of underlying differential equations under a discretization procedure that is useful for numerical implementation. The results are illustrated by a number of examples.

1.1 Introduction

It has been known since E. Noether's fundamental work that conservation laws of differential equations are connected with their symmetry properties [28]. For convenience we present here some well-known results (see also, for example, [1, 3, 18]) for the Lagrangian approach to conservation laws (first integrals). We restrict ourselves to the case with one independent variable.

Let us consider the functional

$$\mathbb{L}(\mathbf{u}) = \int_{t_1}^{t_2} L(t, \mathbf{u}, \dot{\mathbf{u}})\, dt, \tag{1.1}$$

where t is the independent variable, $\mathbf{u} = (u^1, u^2, \ldots, u^n)$ are dependent variables, $\dot{\mathbf{u}} = (\dot{u}^1, \dot{u}^2, \ldots, \dot{u}^n)$ are first-order derivatives and $L(t, \mathbf{u}, \dot{\mathbf{u}})$ is a *first-order* Lagrangian. The functional (1.1) achieves its extremal values when $\mathbf{u}(t)$ satisfies the Euler–Lagrange equations

$$\frac{\delta L}{\delta u^i} = \frac{\partial L}{\partial u^i} - D\left(\frac{\partial L}{\partial \dot{u}^i}\right) = 0, \qquad i = 1, \ldots, n, \tag{1.2}$$

where

$$D = \frac{\partial}{\partial y} + \dot{u}^k \frac{\partial}{\partial u^k} + \ddot{u}^k \frac{\partial}{\partial \dot{u}^k} + \cdots$$

is the total differentiation operator. Here and below we assume summation over repeated indexes. Note that (1.2) are second-order ODEs.

We consider a Lie point transformation group G generated by the infinitesimal operator

$$X = \xi(t, \mathbf{u}) \frac{\partial}{\partial t} + \eta^i(t, \mathbf{u}) \frac{\partial}{\partial u^i} + \cdots , \tag{1.3}$$

where dots mean an appropriate prolongation of the operator to derivatives [5, 21, 29, 30]. The group G is called a variational symmetry of the functional $\mathbb{L}(\mathbf{u})$ if and only if the Lagrangian satisfies [28]

$$X(L) + LD(\xi) = 0, \tag{1.4}$$

where X is the first prolongation, i.e., the prolongation of the vector field X to the first derivatives $\dot{\mathbf{u}}$. We will actually need a weaker invariance condition than given by (1.4). The vector field X is a divergence symmetry of the functional $\mathbb{L}(\mathbf{u})$ if there exists a function $V(t, \mathbf{u}, \dot{\mathbf{u}})$ such that [4] (see also [5, 21, 29])

$$X(L) + LD(\xi) = D(V). \tag{1.5}$$

Generally, (1.5) should hold on the solutions of the Euler–Lagrange equations (1.2).

Noether's theorem [28] can be based on the following Noether-type identity [21], which holds for any vector field and any smooth function $L(t, \mathbf{u}, \dot{\mathbf{u}})$:

$$X(L) + LD(\xi) \equiv (\eta^i - \xi\dot{u}^i)\frac{\delta L}{\delta u^i} + D\left(\xi L + (\eta^i - \xi\dot{u}^i)\frac{\partial L}{\partial \dot{u}^i}\right). \tag{1.6}$$

The theorem states that for a Lagrangian satisfying the condition (1.4) there exists a first integral of the Euler–Lagrange equations (1.2):

$$I = \xi L + (\eta^i - \xi \dot{u}^i)\frac{\partial L}{\partial \dot{u}^i}. \tag{1.7}$$

This result can be generalized [4]: If X is a divergence symmetry of the functional $\mathbb{L}(\mathbf{u})$, i.e., (1.5) is satisfied, then there exists a conservation law

$$I = \xi L + (\eta^i - \xi \dot{u}^i)\frac{\partial L}{\partial \dot{u}^i} - V \tag{1.8}$$

of the corresponding Euler–Lagrange equations.

The strong version of Noether's theorem [21] states that there exists a conservation law of the Euler–Lagrange equations (1.2) in the form (1.7) if and only if the condition (1.4) is satisfied on the solutions of (1.2).

The goal of this chapter is to extend the results presented above to discrete equations in the Lagrangian and Hamiltonian frameworks. We will need to consider canonical Hamiltonian equations before we start to treat their discrete counterparts. It is known that the preservation of first integrals (conservation laws) in numerics is of great importance (see, for example, [19, 31]). Therefore, there is a strong motivation to establish discrete analogs of the conservation properties of the continuous Euler–Lagrange and Hamiltonian equations.

In the next section we will comment on invariance of the Euler–Lagrange equations. In Section 1.3 we will present the Lagrangian formalism for second-order difference equations, which are a discrete analog of the second-order ordinary differential equations. Canonical Hamiltonian equations are considered in Section 1.4. We will develop an analog of Noether's theorem which is based on invariance properties of the action functional, generating canonical Hamiltonian equations. The discrete Hamiltonian equations and their conservation properties are treated in Section 1.5. Section 1.6 presents applications of the theoretical results to a number of examples. Finally Section 1.7 contains concluding remarks.

1.2 Invariance of Euler–Lagrange equations

There exists a relation between the invariance of the Lagrangian function and invariance of the corresponding Euler–Lagrange equations:

Theorem 1.1 ([21, 29]) *If the Lagrangian L is invariant with respect to operator* (1.3)*, i.e., condition* (1.4) *is satisfied, then the Euler–Lagrange equations* (1.2) *are also invariant.*

Remark 1.2 If the Lagrangian L is divergence invariant, i.e., satisfies the condition (1.5), then the Euler–Lagrange equations (1.2) are also invariant. This follows from the fact that full divergences belong to the kernel of variational operators.

Thus, if X is a variational or divergence symmetry of the functional $\mathbb{L}(\mathbf{u})$, it is also a symmetry of the corresponding Euler–Lagrange equations (1.2). The symmetry group of the Euler–Lagrange equations can of course be larger than the group generated by variational and divergence symmetries of the Lagrangian.

It is interesting to establish the necessary and sufficient condition for invariance of the Euler–Lagrange equations. We will need the following lemma:

Lemma 1.3 *For any smooth function $L(t, \mathbf{u}, \dot{\mathbf{u}})$ the following identity holds*

$$\frac{\delta}{\delta u^j}(X(L) + LD(\xi)) \equiv X\left(\frac{\delta L}{\delta u^j}\right) + \left(\frac{\partial \eta^i}{\partial u^j} + \delta_{ij} D(\xi) - \frac{\partial \xi}{\partial u^j}\dot{u}^i\right)\frac{\delta L}{\delta u^i}, \qquad j = 1, \dots, n, \tag{1.9}$$

where the notation δ_{ij} stands for the Kronecker symbol.

Proof The result can be established by a direct computation. □

Theorem 1.1 and Remark 1.2 follow from this lemma. The lemma also provides the *necessary and sufficient* condition for the invariance of the Euler–Lagrange equations:

Theorem 1.4 *The Euler–Lagrange equations* (1.2) *are invariant with respect to a symmetry* (1.3) *if and only if the following conditions are true (on the solutions of the equations):*

$$\left.\frac{\delta}{\delta u^j}(X(L) + LD(\xi))\right|_{\delta L/\delta u^1 = \cdots = \delta L/\delta u^n = 0} = 0, \qquad j = 1, \dots, n. \tag{1.10}$$

Proof The statement follows from the identities of Lemma 1.3. □

Example 1.5 Equation

$$\ddot{u} = \frac{1}{u^2} \tag{1.11}$$

is the Euler–Lagrange equation for the Lagrangian function

$$L(t, u, \dot{u}) = \frac{\dot{u}^2}{2} - \frac{1}{u}.$$

The equation admits symmetries

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = 3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}.$$

The operator X_1 is a symmetry of Lagrangian L and, consequently, a symmetry of (1.11). The symmetry X_2 is not a symmetry of the Lagrangian:

$$X_2(L) + LD(\xi_2) = L.$$

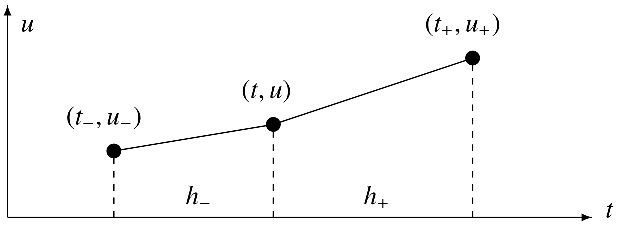
However, it is a symmetry of the equation as follows from Theorem 1.4:

$$\frac{\delta}{\delta u}(X_2(L) + LD(\xi_2))\bigg|_{\delta L/\delta u=0} = \frac{\delta L}{\delta u}\bigg|_{\delta L/\delta u=0} = 0.$$

In the next section we will develop discrete analogs of these results.

1.3 Lagrangian formalism for second-order difference equations

Let us present the results concerning the variational formulation of discrete Euler–Lagrange equations [9–11, 13, 14]. The notations are clear from the following picture:



We consider a finite-difference functional

$$\mathbb{L}_h = \sum_{\Omega} \mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+)h_+, \tag{1.12}$$

defined on some one-dimensional lattice Ω with steplength $h_+ = t_+ - t$. Generally, the lattice can depend on the solution, for example, as

$$\Omega(t, t_-, t_+, \mathbf{u}, \mathbf{u}_-, \mathbf{u}_+) = 0. \tag{1.13}$$

Functional (1.12) must be considered together with lattice (1.13). On different lattices it can have different continuous limits.

Let us take a variation of the difference functional (1.12) along some curve $u^i = \phi_i(t)$, $i = 1, \dots, n$ at some point $(t, \mathbf{u})$. The variation will affect only two terms in the sum (1.12):

$$\mathbb{L}_h = \cdots + \mathcal{L}(t_-, t, \mathbf{u}_-, \mathbf{u})h_- + \mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+)h_+ + \cdots . \tag{1.14}$$

Thus, we get the following expression for the variation of the difference functional

$$\delta \mathbb{L}_h = \frac{\delta \mathcal{L}}{\delta u^i}\delta u^i + \frac{\delta \mathcal{L}}{\delta t}\delta t, \tag{1.15}$$

where $\delta u^i = \phi_i' \delta t$, $i = 1, \ldots, n$ and

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta u^i} &= h_+ \frac{\partial \mathcal{L}}{\partial u^i} + h_- \frac{\partial \mathcal{L}^-}{\partial u^i}, \qquad i = 1, \ldots, n, \\ \frac{\delta \mathcal{L}}{\delta t} &= h_+ \frac{\partial \mathcal{L}}{\partial t} + h_- \frac{\partial \mathcal{L}^-}{\partial t} + \mathcal{L}^- - \mathcal{L}, \end{aligned} \tag{1.16}$$

with $\mathcal{L} = \mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+)$ and $\mathcal{L}^- = \underset{-h}{S}(\mathcal{L}) = \mathcal{L}(t_-, t, \mathbf{u}_-, \mathbf{u})$. For convenience we will use the following total left and right shift operators

$$\underset{-h}{S} f(t, \mathbf{u}) = f(t_-, \mathbf{u}_-), \qquad \underset{+h}{S} f(t, \mathbf{u}) = f(t_+, \mathbf{u}_+)$$

and left and right total difference operators

$$\underset{+h}{D} = \frac{\underset{+h}{S} - 1}{h_+}, \qquad \underset{-h}{D} = \frac{1 - \underset{-h}{S}}{h_-}.$$

Thus, for an arbitrary curve the stationary value of the difference functional is given by a solution of the $n + 1$ equations

$$\frac{\delta \mathcal{L}}{\delta u^i} = 0, \quad i = 1, \ldots, n, \qquad \frac{\delta \mathcal{L}}{\delta t} = 0, \tag{1.17}$$

called *global extremal* equations. These equations represent the entire difference scheme and could be called “the discrete Euler–Lagrange system.” They can be interpreted as a three-point difference scheme of the form

$$\begin{aligned} &F_i(t, t_-, t_+, \mathbf{u}, \mathbf{u}_-, \mathbf{u}_+) = 0, \qquad i = 1, \ldots, n, \\ &\Omega(t, t_-, t_+, \mathbf{u}, \mathbf{u}_-, \mathbf{u}_+) = 0. \end{aligned}$$

Here the first n equations are approximations of differential equations (1.2) and the last equation provides a lattice, on which these approximations are considered. In the continuous limit the lattice equation vanishes (turns into an identity like $0 = 0$). Given two points, for instance $(t, \mathbf{u})$ and $(t_-, \mathbf{u}_-)$, we can calculate $(t_+, \mathbf{u}_+)$.

Note that the variational equations (1.17) can be obtained by the action of discrete variational operators

$$\frac{\delta}{\delta u^i} = \frac{\partial}{\partial u^i} + \underset{-h}{S} \frac{\partial}{\partial u^i_+}, \qquad i = 1, \ldots, n, \tag{1.18}$$

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \underset{-h}{S} \frac{\partial}{\partial t_+} \tag{1.19}$$

on the discrete elementary action $\mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+)h_+$.

Now let us consider a variation of the functional (1.12) along the orbit of a group generated by the operator (1.3). Then, we have $\delta t = \xi \delta a$, $\delta u^i = \eta^i \delta a$, $i = 1, \ldots, n$, where δa is the variation of the group parameter. A stationary value of the difference functional (1.12) along the flow generated by this vector field is given by the equation

$$\eta^i \frac{\delta \mathcal{L}}{\delta u^i} + \xi \frac{\delta \mathcal{L}}{\delta t} = 0, \tag{1.20}$$

which depends explicitly on the coefficients of the generator. This equation is called a *quasiextremal* equation. If we have a Lie algebra of vector fields of dimension $n + 1$ or more, then the stationary value of difference functional (1.12) along the entire flow will be achieved on the intersection of the solutions of all quasiextremal equations of the type (1.20), i.e., the system of equations (1.17).

Remark 1.6 Sometimes it is convenient to consider the variational equations (1.17) in a modified form

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial u^i} + \frac{h_-}{h_+} \frac{\partial \mathcal{L}^-}{\partial u^i} = 0, \qquad i = 1, \ldots, n, \\ &\frac{\partial \mathcal{L}}{\partial t} + \frac{h_-}{h_+} \frac{\partial \mathcal{L}^-}{\partial t} - \underset{+h}{D}(\mathcal{L}^-) = 0, \end{aligned} \tag{1.21}$$

obtained on dividing by h_+.

Let us consider a Lie group of point transformations, generated by a vector field (1.3). When acting on discrete equations and functionals, a vector field must be prolonged to variables at other points of the lattice. The prolongation is obtained by shifting the coefficients to the corresponding points. For three-point schemes we have

$$\begin{aligned} X = \xi \frac{\partial}{\partial t} + \xi_- \frac{\partial}{\partial t_-} + \xi_+ \frac{\partial}{\partial t_+} + \eta^i \frac{\partial}{\partial u^i} + \eta^i_- \frac{\partial}{\partial u^i_-} + \eta^i_+ \frac{\partial}{\partial u^i_+} \\ + (\xi_+ - \xi) \frac{\partial}{\partial h_+} + (\xi - \xi_-) \frac{\partial}{\partial h_-}, \end{aligned} \tag{1.22}$$

where coefficients are given as follows

$$\xi_- = \xi(t_-, \mathbf{u}_-), \qquad \eta^i_- = \eta^i(t_-, \mathbf{u}_-), \qquad \xi_+ = \xi(t_+, \mathbf{u}_+), \qquad \eta^i_+ = \eta^i(t_+, \mathbf{u}_+).$$

The infinitesimal invariance condition for the functional (1.12) on the lattice (1.13) is given by two equations [9–11, 14]:

$$X(\mathcal{L}) + \mathcal{L} \underset{+h}{D}(\xi) \Big|_{\Omega=0} = 0, \qquad X(\Omega)\Big|_{\Omega=0} = 0, \tag{1.23}$$

which are valid on the lattice (1.13). Generally, the lattice is provided by the global extremal equations (1.17). Therefore, we need to require their invariance to consider the invariance of the functional.

A useful operator identity, valid for any Lagrangian $\mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+)$ and any vector field X is [9, 11]

$$X(\mathcal{L}) + \mathcal{L} \underset{+h}{D}(\xi) \equiv \xi\left(\frac{\partial \mathcal{L}}{\partial t} + \frac{h_-}{h_+}\frac{\partial \mathcal{L}^-}{\partial t} - \underset{+h}{D}(\mathcal{L}^-)\right)$$
$$+ \eta^i\left(\frac{\partial \mathcal{L}}{\partial u^i} + \frac{h_-}{h_+}\frac{\partial \mathcal{L}^-}{\partial u^i}\right) + \underset{+h}{D}\left(h_-\eta^i\frac{\partial \mathcal{L}^-}{\partial u^i} + h_-\xi\frac{\partial \mathcal{L}^-}{\partial t} + \xi\mathcal{L}^-\right). \quad (1.24)$$

The identity is a discrete analog of Noether identity (1.6) and can be called *the discrete Noether identity*. From this relation we obtain the following discrete analog of Noether's theorem.

Theorem 1.7 ([9, 11, 14]) *The global extremal equations* (1.17)*, invariant under the Lie group G of local point transformations generated by vector fields X of the form* (1.3)*, possess a first integral*

$$\mathcal{I} = h_-\eta^i\frac{\partial \mathcal{L}^-}{\partial u^i} + h_-\xi\frac{\partial \mathcal{L}^-}{\partial t} + \xi\mathcal{L}^- \quad (1.25)$$

if and only if the Lagrangian density $\mathcal{L}$ *is invariant with respect to the same group on the solutions of* (1.17).

Remark 1.8 If the Lagrangian density $\mathcal{L}$ is divergence invariant under Lie group G of local point transformations, i.e.,

$$X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi) = \underset{+h}{D}(V) \quad (1.26)$$

for some function $V(t, \mathbf{u})$, then each element X of the Lie algebra corresponding to group G provides us with a first integral of the global extremal equations (1.17), namely

$$\mathcal{I} = h_-\eta^i\frac{\partial \mathcal{L}^-}{\partial u^i} + h_-\xi\frac{\partial \mathcal{L}^-}{\partial t} + \xi\mathcal{L}^- - V. \quad (1.27)$$

Remark 1.9 In a particular case when the discrete Lagrangian is invariant with respect to time translations, i.e., $\mathcal{L} = \mathcal{L}(h_+, \mathbf{q}, \mathbf{q}_+)$, where $h_+ = t_+ - t$ is the step size, there is a conservation of energy

$$\mathcal{E} = -\mathcal{L}^- - h_-\frac{\partial \mathcal{L}^-}{\partial h_-} = -\mathcal{L} - h_+\frac{\partial \mathcal{L}}{\partial h_+}.$$

In this case we get symplectic-momentum-energy preserving variational integrators [22].

It has been shown elsewhere [9–11], that if the functional (1.12) is invariant or divergence invariant under some group G, then the global extremal equations (1.17) are also invariant with respect to G:

Theorem 1.10 *If the Lagrangian $\mathcal{L}$ is invariant with respect to the operator* (1.3)*, then the global extremal equations* (1.17) *are also invariant.*

Remark 1.11 If the Lagrangian $\mathcal{L}$ is divergence invariant, then the global extremal equations (1.17) are also invariant. This follows from the fact that total finite differences belong to the kernel of discrete variational operators.

As in the continuous case, the global extremal equations can be invariant with respect to a larger group than the corresponding Lagrangian.

Now we are in a position to establish the necessary and sufficient condition for the invariance of global extremal equations. We will obtain new identities and a new theorem.

Lemma 1.12 *The following identities hold for any smooth function $\mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+)$:*

$$\frac{\delta}{\delta u^j}\left(\left(X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi)\right)h_+\right) \equiv X\left(\frac{\delta \mathcal{L}}{\delta u^j}\right) + \frac{\partial \eta^i}{\partial u^j}\frac{\delta \mathcal{L}}{\delta u^i} + \frac{\partial \xi}{\partial u^j}\frac{\delta \mathcal{L}}{\delta t}, \qquad j = 1, \ldots, n, \tag{1.28}$$

$$\frac{\delta}{\delta t}\left(\left(X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi)\right)h_+\right) \equiv X\left(\frac{\delta \mathcal{L}}{\delta t}\right) + \frac{\partial \eta^i}{\partial t}\frac{\delta \mathcal{L}}{\delta u^i} + \frac{\partial \xi}{\partial t}\frac{\delta \mathcal{L}}{\delta t}. \tag{1.29}$$

Proof The identities can be verified directly. □

The lemma allows us to obtain not only the sufficient (Theorem 1.10) but also the necessary and sufficient condition for the invariance of the global extremal equations.

Theorem 1.13 *The global extremal equations* (1.17) *are invariant with respect to a symmetry* (1.3) *if and only if the following conditions are true (on the solutions of the equations):*

$$\frac{\delta}{\delta u^j}\left(\left(X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi)\right)h_+\right)\bigg|_{(1.17)} = 0, \qquad j = 1, \ldots, n, \tag{1.30}$$

$$\frac{\delta}{\delta t}\left(\left(X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi)\right)h_+\right)\bigg|_{(1.17)} = 0. \tag{1.31}$$

Proof The statement follows from identities of Lemma 1.12. □

Many examples of applications of the discrete version of Noether's theorem in Lagrangian framework can be found in [14]. It should be noted that the discrete Lagrangian formalism and the corresponding Noether's theorem are not restricted to ordinary equations. They can also be used for discretizations of partial differential equations [6].

We note that there exists an alternative approach to conservation laws of discrete equations on fixed meshes based on direct methods [20].

1.4 Hamiltonian formalism for differential equations

In this chapter we will also present the Hamiltonian formalism for discrete Hamiltonian equations. Before that we consider the canonical Hamiltonian equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad i = 1, \ldots, n \tag{1.32}$$

and rewrite results concerning their invariance and conservation properties in a nonstandard way, that provides us with a simple "translation" of the Lagrangian formalism into the Hamiltonian one. We also present a new criterion (Theorem 1.23) for the invariance of the Hamiltonian equations.

1.4.1 Canonical Hamiltonian equations

It is well known that canonical Hamiltonian equations (1.32) can be obtained by the variational principle from the action functional

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}^i - H(t, \mathbf{q}, \mathbf{p}))\, \mathrm{d}t = 0 \tag{1.33}$$

in the phase space $(\mathbf{q}, \mathbf{p})$, where $\mathbf{q} = (q^1, q^2, \ldots, q^n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ [17, 27]. Let us note that the canonical Hamiltonian equations (1.32) can be derived by action of the variational operators

$$\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i} - D\frac{\partial}{\partial \dot{p}_i}, \qquad i = 1, \ldots, n, \tag{1.34}$$

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D\frac{\partial}{\partial \dot{q}^i}, \qquad i = 1, \ldots, n, \tag{1.35}$$

where D is the operator of total differentiation with respect to time

$$D = \frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k} + \dot{p}_k \frac{\partial}{\partial p_k} + \cdots,$$

on the function

$$p_i \dot{q}^i - H(t, \mathbf{q}, \mathbf{p}).$$

As an analog of Lagrangian elementary action $L\,\mathrm{d}t$ [21, 29] we consider Hamiltonian elementary action [12], namely

$$p_i\, \mathrm{d}q^i - H(t, \mathbf{q}, \mathbf{p})\, \mathrm{d}t, \tag{1.36}$$

and investigate its invariance with respect to a point transformation group generated by an operator

$$X = \xi(t, \mathbf{q}, \mathbf{p})\frac{\partial}{\partial t} + \eta^i(t, \mathbf{q}, \mathbf{p})\frac{\partial}{\partial q^i} + \zeta_i(t, \mathbf{q}, \mathbf{p})\frac{\partial}{\partial p_i}. \tag{1.37}$$