1
Differential geometry

A space–time can be described naively as a four-dimensional space in which a metric is defined by means of a line element ds, whose square is a non-degenerate indefinite quadratic differential form,

$$ds^2 = g_{ij}dx^i dx^j \quad (g_{ij} = g_{ji}). \tag{1.0.1}$$

The summation convention is used here, and the g_{ij} are functions of $x = (x^1, x^2, x^3, x^4)$. The signature of the form is -2, so that

$$ds^2 = (l_{1j}dx^i)^2 - (l_{2j}dx^i)^2 - (l_{3j}dx^j)^2 - (l_{4j}dx^j)^2,$$

where the linear differential forms $l_{ij}dx^j$ are linearly independent. (Many authors take the signature of ds^2 to be 2.) All coordinate systems that are related to each other by sufficiently differentiable coordinate transformations are considered as equivalent frames of reference, and as ds^2 is invariant, the g_{ij} transform as the components of a symmetric covariant tensor.

Denote the determinant of the g_{ij} by g, and the inverse of the matrix (g_{ij}) by (g^{ij}). It is well known that the linear second-order differential operator

$$\Box u = |g|^{-\frac{1}{2}} \frac{\partial}{\partial x^i}\left(|g|^{\frac{1}{2}} g^{ij} \frac{\partial u}{\partial x^j}\right) \tag{1.0.2}$$

is invariant under coordinate transformations. For the metric of special relativity,

$$ds^2 = (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - (dx^4)^2,$$

the equation $\Box u = 0$ is the ordinary wave equation,

$$\left(\left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \left(\frac{\partial}{\partial x^3}\right)^2 - \left(\frac{\partial}{\partial x^4}\right)^2\right) u = 0.$$

One can therefore regard the differential equation $\Box u = 0$ as a wave equation in the space–time with the metric (1.0.1). One can also consider the more general equation

$$Pu = \Box u + a^i \frac{\partial u}{\partial x^i} + bu = f, \tag{1.0.3}$$

where the a^i are the components of a vector field, b is a scalar field, and f (the 'source term') is a given function. It is this equation which will here be called a wave equation, or a scalar wave equation. There are, also, analogous vector and tensor wave equations. Such equations occur in general relativity, where they govern the propagation of test fields, small disturbances whose effect on the space–time background can be neglected.

Because of (1.0.2), the equation (1.0.3) is, in a fixed coordinate system, an equation of hyperbolic type. Hyperbolic equations occur in many contexts, and usually govern wave propagation phenomena. Clearly, it is natural to associate a space–time with such an equation. This has the formal advantage that it facilitates coordinate transformations. But, in fact, the connection is much deeper, because the characteristics of the equation (1.0.3) are the null hypersurfaces of the metric (1.0.1).

The description of space–time given above is inadequate, and it is better to define a space–time from the outset as a differentiable manifold with a Lorentzian metric. This chapter is an outline of the relevant concepts and results in differential geometry.

The first section outlines the basic definition and concepts, such as differentiable structure, tangent and cotangent spaces and bundles, tensor fields, and metrics. The second section deals in rather more detail with geodesics and the exponential map. It includes a proof of Whitehead's theorem on the existence of geodesically convex domains, and the derivation of the properties of the square of the geodesic distance between points in a geodesically convex domain. This material will be of particular importance later on. In the last section, exterior forms are introduced, and the integral of an exterior form of maximal degree is defined. For a manifold with a metric, a form which can serve as an invariant volume element is defined; this will be constantly used in the sequel. The section ends with a statement of the divergence theorem.

The reader's attention is also drawn to the appendix to this book, which summarizes some elementary topological definitions and results.

1.1 Differentiable manifolds

An *n-dimensional manifold* M is a topological space, every point of which has a neighbourhood that is homeomorphic to an open set in $\mathbf{R}^n$.

If Ω is such a neighbourhood, and $\pi: \Omega \to \mathbf{R}^n$ the homeomorphism in question, then π sends every point $p \in \Omega$ to a point

$$\pi p = x = (x^1, \ldots, x^n) \in \mathbf{R}^n;$$

the x^i are *local coordinates* of p, Ω is a *coordinate neighbourhood*, π is a *coordinate system*, and the pair (Ω, π) will be called a *chart*. (A chart at p will be any chart (Ω, π) such that $p \in \Omega$.) Suppose that (Ω, π) and $(\tilde{\Omega}, \tilde{\pi})$ are charts, and that $\Omega \cap \tilde{\Omega}$ is not empty. Then the respective local coordinates $x = \pi p$ and $\tilde{x} = \tilde{\pi} p$ of a point $p \in \Omega \cap \tilde{\Omega}$ are related by the map

$$\tilde{\pi} \circ \pi^{-1}: \pi(\Omega \cap \tilde{\Omega}) \to \tilde{\pi}(\Omega \cap \tilde{\Omega}), \tag{1.1.1}$$

which is called a *coordinate transformation*, and is a homeomorphism between open sets in $\mathbf{R}^n$.

A C^∞ *structure* on a manifold M is an indexed family of charts $\{\Omega_\nu, \pi_\nu\}$ that has the following properties:

(i) $\{\Omega_\nu\}$ is a covering of M, $\bigcup \Omega_\nu = M$.

(ii) The coordinate transformations $\pi_\nu \circ \pi_\mu^{-1}$ are infinitely differentiable.

(iii) If (Ω, π) is a chart such that $\pi_\nu \circ \pi^{-1}$ is infinitely differentiable for all ν, then (Ω, π) is a chart belonging to the structure.

A manifold with a C^∞ structure is called a C^∞ manifold. If one drops condition (iii), one obtains a covering of M by C^∞ charts, which can be extended to a C^∞ structure; this extension is unique. If (ii) is replaced by the requirement that the coordinate transformations are to be C^k, which means that they have continuous derivatives of all orders less than or equal to k, one has a C^k structure. A C^∞ structure is also a covering of M by C^k charts; hence it has a unique extension to a C^k structure. We shall usually work with C^∞ manifolds, but occasionally extend the C^∞ structure to a C^k structure with finite k.

If $f(p)$ is a function on M and (Ω, π) is a chart, then

$$f \circ \pi^{-1}(x) = f(\pi^{-1}x)$$

is a function on $\pi\Omega \subset \mathbf{R}^n$. (In this chapter, functions are assumed to be real-valued; in subsequent chapters, complex-valued functions will also occur.) If $f \circ \pi^{-1}$ is infinitely differentiable for all charts, then we say that $f \in C^\infty$ or that $f \in C^\infty(M)$; it is enough for this to hold in each one of a family of charts covering M. The class $C^k(M)$ is defined similarly. If $D \subset M$ is an open set, and f is a function defined on D, then $f \in C^\infty(D)$ (respectively, $f \in C^k(D)$) means that $f \circ \pi^{-1}: \pi(\Omega \cap D) \to \mathbf{R}$ is C^∞ (respectively, C^k) for all charts (Ω, π) such that Ω meets D.

The *support* of a function $f: M \to \mathbf{R}$ is the closure of the set $\{p; f(p) \neq 0\}$; it is denoted by $\operatorname{supp} f$. If $D \subset M$ is an open set, then $C_0^\infty(D)$ is the class of all $f \in C^\infty(D)$ whose supports are compact subsets of D; $C_0^k(D)$ is defined in the same way.

An open covering $\{\Omega_\nu\}$ of M is called *locally finite* if every compact set meets only a finite number of the Ω_ν. A topological space is called *paracompact* if every open covering has a refinement that is locally finite; C^∞ manifolds will be assumed, throughout, to be paracompact. A *partition of unity* is an indexed family of functions $\phi_\nu \in C_0^\infty(M)$ whose supports are a locally finite covering of M, and which is such that

$$0 \leqslant \phi_\nu \leqslant 1, \quad \sum_\nu \phi_\nu = 1. \tag{1.1.2}$$

Note that, at each point of M, at least one of the ϕ_ν is non-zero, and all but a finite number of the ϕ_ν are zero. It can be shown that, given a covering $\{\Omega_j\}$ of M by open sets, there is a partition of unity $\{\phi_\nu\}$ subordinated to this covering. This means that, for each j, there is a ν such that $\operatorname{supp} \phi_\nu \subset \Omega_j$. If the covering is locally finite, and the Ω_j are relatively compact, then one can choose a partition of unity $\{\phi_j\}$ such that $\operatorname{supp} \phi_j \subset \Omega_j$ for each j. This will be the usual situation in the sequel, and the Ω_j will generally be coordinate neighbourhoods.

Let M' and M be C^∞ manifolds, of dimensions m and n respectively, and let g be a map $M' \to M$. Such a map is said to be C^∞ if the map $\pi \circ g \circ \pi'^{-1}: \pi'\Omega' \to \mathbf{R}^n$ is C^∞ for all pairs of charts (Ω', π') and (Ω, π), in M' and M respectively, such that $g\Omega' \subset \Omega$. Let p' be a point in M', $p = gp'$ its image under g in M, and (Ω', π'), (Ω, π) be charts at p' and p respectively, such that $\pi\Omega' \subset \Omega$. The rank of g at p' is then, by definition, the rank of the Jacobian matrix $D(\pi \circ g \circ \pi'^{-1})$ at $\pi'^{-1}p'$; it is evidently independent of the choice of the charts (Ω', π') and (Ω, π). The map g is called an *imbedding* if it is one-to-one (injective) and its rank is equal to m, the dimension of M', at all points of M'. If g is an imbedding and onto (surjective), then it is a *diffeomorphism*; one then has, necessarily, $m = n$. With the obvious modifications, one can also define maps, imbeddings, and diffeomorphisms, of class C^k.

An m-dimensional *sub-manifold* of M, where $m \leqslant n$, is a set $M' \subset M$ which can be covered by coordinate charts (Ω, π) such that

$$\Omega \cap M' = \{p; p \in \Omega, \quad x = \pi p, \quad x^{m+1} = \ldots = x^n = 0\}; \tag{1.1.3}$$

if $m = n$, M' is just an open set in M. One can consider M' as an m-dimensional manifold, with the differentiable structure inherited from that of M; the inclusion map $M' \to M$ is then obviously an imbedding. Conversely, the image of an imbedding is always a submanifold. The non-negative integer $n-m$ is the *codimension* of M'. If $m = 1$, M' will be called a *curve*; if $m = n-1$, a *hypersurface*; and if $1 < m < n-1$, it will be sometimes be called an *m-surface*. A submanifold M' of codimension k can be specified by giving k C^∞ functions $S_1(p), \ldots, S_k(p)$ and setting $M' = \{p; S_1(p) = 0, \ldots, S_k(p) = 0\}$, provided that the rank of the map $M \to \mathbf{R}^k$ which sends $p \in M$ to

$$(S_1(p), \ldots, S_k(p)) \in \mathbf{R}^k$$

is equal to k at all points of M'; for a hypersurface ($k = 1$) this means that the gradient of $S(p)$ must be non-zero at all points of $\{p; S(p) = 0\}$. (The gradient of a function is defined below.) Locally, any submanifold can be given in this form, by (1.1.3).

So far, we have only considered unbounded manifolds. A *manifold-with-boundary* is defined in the same way as an unbounded manifold, except that the sets $\pi\Omega$ are only required to be open subsets of the closed half-space $\overline{\mathbf{R}^n_+} = \{x; x \in \mathbf{R}^n, x^n \geqslant 0\}$. If $\pi_\nu(\Omega_\nu \cap \Omega_\mu)$ contains points in $\partial\overline{\mathbf{R}^n_+} = \{x; x \in \mathbf{R}^n, x^n = 0\}$, condition (ii) above must be understood to mean that the coordinate transformation $\pi_\nu \circ \pi_\mu^{-1}$ can be extended to a diffeomorphism between open sets in $\mathbf{R}^n$. It is obvious that, for any point $p \in M$, πp is either in $\mathbf{R}^n_+ = \{x; x \in \mathbf{R}^n, x^n > 0\}$ for all charts at p, or in $\partial\overline{\mathbf{R}^n_+}$. In the former case, p is an interior point, in the latter, it is a boundary point. The set of boundary points is the *boundary* ∂M of M; it is an $(n-1)$-dimensional C^∞ manifold, with the differentiable structure inherited from that of M.

An n-dimensional manifold M can always be imbedded in a Euclidean space $\mathbf{R}^N$, where N is sufficiently large; by Whitney's embedding theorem, one can take $N = 2n+1$. Considering $\mathbf{R}^N$ as a vector space, one can see intuitively that the tangent vectors to curves that go through a fixed point $p \in M$ form an n-dimensional vector space, the tangent space at p. For an intrinsic definition, we consider parametrized curves through p. By this we mean a C^1 map $t \to f(t)$ of an interval $I_\delta = (-\delta, \delta) \in \mathbf{R}$ into M, such that $f(0) = p$. If (Ω, π) is a chart at p, and δ is sufficiently small, then $f(I_\delta) \in \Omega$, and $t \to \pi \circ f(t)$ is a curve in $\mathbf{R}^n$ that has a tangent vector

$$\xi = \frac{d}{dt}(\pi \circ f(t))|_{t=0} \tag{1.1.4}$$

at the point $\pi \circ f(0) \in \mathbf{R}^n$. Let $(\tilde{\Omega}, \tilde{\pi})$ be another chart at p and suppose that $f(I_\delta) \in \tilde{\Omega}$; we obtain another Euclidean tangent vector $\tilde{\xi}$, to the curve $t \to \tilde{\pi} \circ f(t)$ at $\tilde{\pi} \circ f(0)$. If one reads ξ and $\tilde{\xi}$ as column vectors, and the Jacobian $D(\tilde{\pi} \circ \pi^{-1})$ of the coordinate transformation as an $n \times n$ matrix, then it follows from the chain rule for partial derivatives that ξ and $\tilde{\xi}$ are related by the law of contravariance,

$$\tilde{\xi} = D(\tilde{\pi} \circ \pi^{-1})|_{\pi p}\,\xi. \tag{1.1.5}$$

Guided by this relation, we now consider the set of triples (Ω, π, ξ), where (Ω, π) is a coordinate chart at p and ξ is a vector in $\mathbf{R}^n$. It is easily verified that the relation

$$(\Omega, \pi, \xi) \cong (\tilde{\Omega}, \tilde{\pi}, \tilde{\xi}) \quad \text{if} \quad \tilde{\xi} = D(\tilde{\pi} \circ \pi^{-1})|_{\pi p}\,\xi$$

is an equivalence relation; a *tangent vector* v at p is then, by definition, an equivalence class with respect to this relation. One usually writes $\xi = \pi_* v$ for the components of ξ. The tangent vectors at p obviously form an n-dimensional vector space over $\mathbf{R}$; addition, and multiplication by a number, can be defined, componentwise, in any coordinate system. This is the *tangent space* at p, and is denoted by TM_p. It follows from (1.1.4) that a parametrized C^1 curve that goes through p determines a tangent vector at p, which will sometimes be denoted by $f'(0)$ or by $(d/dt)f(0)$. Conversely, it is evident that any tangent vector at p can be considered to be the tangent vector to some parameterized curve through p.

The set of all tangent vectors to M can be made into a C^∞ manifold, which is called the *tangent bundle*, and denoted by TM. If $v \in TM$, then it is in one and only one TM_p; let Π denote the projection, which is the map $v \to p$. Let (Ω, π) be a coordinate chart, and suppose that $v \in \Pi^{-1}\Omega$. Assign to v the coordinates $\kappa v = (\pi \circ \Pi v, \pi_* v) = (x, \xi)$, say. One can easily show that there is a unique topology for TM such that, for all charts in the C^∞ structure of M, κ is a homeomorphism on an open set in $\mathbf{R}^n \times \mathbf{R}^n$. If $(\tilde{\Omega}, \tilde{\pi})$ is another chart, and $\Omega \cap \tilde{\Omega}$ is not empty, then it follows from (1.1.5) that the coordinate transformation in TM is

$$\tilde{x} = \tilde{\pi} \circ \pi^{-1} x, \quad \tilde{\xi} = D(\tilde{\pi} \circ \pi^{-1})\,\xi \tag{1.1.6}$$

so that the tangent bundle has a C^∞ structure.

A *vector field* can now be defined as a *cross-section* of TM; this is a map $V: M \to TM$ such that $\pi \circ V$ is the identity. In local coordinates, this just means that $V \circ \pi^{-1} x = (x, \xi(x))$. In view of what has already been said about maps of one C^∞ manifold into another one, it is evident how C^k and C^∞ vector fields are defined. In local coordinates, the components $V^i(x)$ of a C^∞ vector field are, of course, C^∞ functions of x, and transform by contravariance.

The *cotangent space* T^*M_p at a point $p \in M$ is the dual of TM_p; it consists of all linear maps w: $TM_p \to \mathbf{R}$, made into an n-dimensional vector space over $\mathbf{R}$ in the natural way. Its elements are called *covectors*. The value of the covector w at $v \in TM_p$ will be denoted by $\langle w, v\rangle$, and called the *scalar product* of w and v. An important example of a covector is the *gradient* of a C^1 function at p. Let u be a C^1 function, defined on a neighbourhood of p, and let $v \in TM_p$. One can always find a parameterized curve $t \to f(t)$ such that $f(0) = p$, and $f'(0) = v$. The composite function $t \to u \circ f(t)$ is a C^1 function, on some interval $(-\delta, \delta) \in \mathbf{R}$, and

$$f'(0) = v \to du(p) = \frac{d}{dt}(u \circ f(t))|_{t=0}$$

is a linear form on TM_p, and so a covector. We denote this map by $\operatorname{grad} u(p)$ or $\nabla u(p)$, and write

$$du(p) = \langle \operatorname{grad} u(p), v\rangle \equiv \langle \nabla u(p), v\rangle. \tag{1.1.7}$$

Let (Ω, π) be a coordinate chart at p. The coordinate curves are the images, under π^{-1}, of the coordinate lines in $\mathbf{R}^n$ at πp,

$$x^i = (\pi p)^i + \delta^i_j t \quad (i, j = 1, \ldots, n),$$

where the δ^i_j are the Kronecker deltas. The tangent vectors $e_{(i)}$ to these curves are a basis of TM_p, associated in a natural way with (Ω, π); we have already used this basis, for if $\xi = \pi_* v$ then the ξ^i are just the components of v with respect to the basis $\{e_{(i)}\}_{1 \leqslant i \leqslant n}$. The dual basis of $\{e_{(i)}\}_{1 \leqslant i \leqslant n}$ consists of the covectors $e^{*(j)}$ such that $\langle e^{*(j)}, e_{(i)}\rangle = \delta^j_i$. So, if one writes

$$v = \xi^i e_{(i)}, \quad w = \omega_i e^{*(i)}, \tag{1.1.8}$$

using the summation convention, then one obtains the usual identity

$$\langle w, v\rangle = \omega_i \xi^i. \tag{1.1.9}$$

In particular, it follows from (1.1.7) and

$$du(p) = \frac{d}{dt}(u \circ \pi^{-1}) \circ (\pi \circ f)|_{t=0}$$

that

$$du = \langle \operatorname{grad} u(p), v\rangle = \xi^i \frac{\partial}{\partial x^i} u \circ \pi^{-1}(x), \tag{1.1.10}$$

which is the classical form of the gradient.

The components ω_i of a covector transform by covariance, as can immediately be inferred from (1.1.9) and (1.1.5); if one reads $\{\omega_i\}_{1 \leqslant i \leqslant n}$ as a row matrix, this is

$$\tilde{\omega} = \omega D\pi \circ \tilde{\pi}^{-1}|_{\tilde{\pi} p}. \tag{1.1.11}$$

We shall frequently use classical notation, which, although ambiguous, is more convenient in computations. By an abuse of language, $u\circ\pi^{-1}(x), u\circ\tilde{\pi}^{-1}(\tilde{\pi}), \ldots$ are denoted by $u(x), u(\tilde{x}), \ldots$; a tangent vector is written as a differential $dx = (dx^1, \ldots, dx^n)$, so that (1.1.10) assumes the familiar form $du = (\partial u/\partial x^i)\,dx^i$. Of course, dx^i has two distinct meanings; it may be a component of a tangent vector, or a covector, which is the gradient of a coordinate function $(\pi p)^i = x^i$. But no confusion is likely to arise in practice if this is borne in mind.

The *cotangent bundle* is constructed like the tangent bundle; it is a C^∞ manifold T^*M of dimension $2n$. Its coordinate transformations are obtained by combining the coordinate transformations of M and the law of covariance (1.1.11). A covector field is then a cross-section of T^*M. (A covector is of course the same as a 'classical' covariant vector.) A gradient field is an example of a covector field.

Suppose that S is a sub-manifold of M, of dimension $m < n$. The tangent space to S at a point $p \in S$, TS_p, is obviously the subspace of TM_p that is spanned by the tangent vectors to parametrized curves through p that are in S. The annihilator of TS_p is the subspace of T^*M_p defined by

$$N_p = \{w;\, w \in T^*M_p, \langle w, v\rangle = 0 \quad \text{for all} \quad v \in TS_p\},$$

and is, clearly, the *normal* to S. If S is a hypersurface ($m = n-1$), then N_p is one-dimensional; if this hypersurface is given as $\{p;\, u(p) = 0\}$ then N_p is the one-dimensional subspace of T^*M_p which contains $\operatorname{grad} u(p)$. (Recall that, by definition, $\operatorname{grad} u(p) \neq 0$ at all points of a hypersurface $S = \{p;\, u(p) = 0\}$.)

Covectors have been introduced as linear forms on TM_p. By duality, vectors can also be considered as linear forms on T^*M_p. A *tensor of type* (r, s) is a multilinear form

$$S(w_{(1)}, \ldots, w_{(r)}, v_{(1)}, \ldots, v_{(s)}),$$

that is to say, a map

$$S\colon \underbrace{T^*M_p \times \ldots \times T^*M_p}_{r \text{ factors}} \times \underbrace{TM_p \times \ldots \times TM_p}_{s \text{ factors}} \to \mathbf{R}$$

that is linear when restricted to any one of the factors TM_p or T^*M_p. Tensors of fixed type are again a vector space over $\mathbf{R}$, addition and multiplication by a number being carried out with the multilinear forms. To define the *tensor product* of a tensor S of type (r, s) and a tensor T of type (k, l), one simply puts

$$\begin{aligned} S \otimes T(&\omega_{(1)}, \ldots, \omega_{(r+k)}, v_{(1)}, \ldots, \nu_{(s+l)}) \\ &= S(\omega_{(1)}, \ldots, w_{(r)}, \nu, \ldots, \nu_{(s)})\, T(\omega_{(r+1)}, \ldots, w_{(r+k)}, \nu_{(s+1)}, \ldots, \nu_{(s+l)}). \end{aligned}$$

As tensors of type $(1,0)$ are vectors, and tensors of type $(0,1)$ are covectors, tensors of all types can be built up by forming tensor products of vectors and covectors. So, if (Ω,π) is a chart at p, and $\{e_{(i)}\}_{1\leqslant i\leqslant n}$, $\{e^{*(i)}\}_{1\leqslant i\leqslant n}$ are the associated dual bases of TM_p and T^*M_p respectively, then a basis of the vector space of tensors of type (r,s) is provided by the tensors $e_{(i_1)}\otimes\ldots\otimes e_{(i_r)}\otimes e^{*(j_1)}\otimes\ldots\otimes e^{*(j_s)}$, and a tensor T of type (r,s) can be written as

$$T = T^{i_1\ldots i_r}{}_{j_1\ldots j_s}\, e_{(i_1)}\otimes\ldots\otimes e_{(i_r)}\otimes e^{*(j_1)}\otimes\ldots\otimes e^{*(j_s)},$$

with the usual transformation law for the tensor components,

$$\tilde{T}^{i_1\ldots i_r}{}_{j_1\ldots j_s} = \frac{\partial\tilde{x}^{i_1}}{\partial x^{k_1}}\cdots\frac{\partial\tilde{x}^{i_r}}{\partial x^{k_s}}T^{k_1\ldots k_r}{}_{l_1\ldots l_s}\frac{\partial x^{l_1}}{\partial\tilde{x}^{j_1}}\cdots\frac{\partial x^{l_s}}{\partial\tilde{x}^{j_s}}.$$

Finally, tensor fields are defined as cross-sections of the appropriate tensor bundle over M. The scalar product is a tensor of type $(1,1)$, whose components, at all points, and in all coordinate systems, are the Kronecker deltas.

A tensor of type $(0,2)$ at a point $p\in M$, $G(v,v')$ is called symmetric if $G(v,v') = G(v',v)$ for all tangent vectors v, v' at p; it is called non-degenerate if $G(v,v') = 0$ for all $v'\in TM_p$ implies that $v = 0$. In local coordinates, this means that the components are a symmetric non-singular matrix (g_{ij}). A *metric*, on a C^∞ manifold M, is a non-degenerate symmetric C^∞ tensor field $G(p;v,v')$ of type $(0,2)$. We define the *signature* of G at a point p to be the sum of the signs of the eigenvalues of the matrix $(g_{ij}(x))$, where the x^i are the local coordinates of p. By the law of inertia for quadratic forms, the signature is independent of the choice of the local coordinate system, and it follows from continuity that it is the same at all points of M, if M is connected. If the signature is equal to n, the dimension of the manifold, so that $G(p;v,v')$ is positive definite, then the metric is *Riemannian*. Otherwise, it is called *pseudo-Riemannian* (unless it is equal to $-n$, when one can replace G by $-G$). If the modulus of the signature of the metric is $n-2$, then the metric is called *hyperbolic* or *Lorentzian*. A space–time is a C^∞ manifold with a Lorentzian metric; we shall always take the signature to be $2-n$.

Any C^∞ manifold admits a Riemannian metric, and any non-compact C^∞ manifold admits a Lorentzian metric.

At a point $p\in M$, the metric furnishes an *inner product* for TM_p, which is just $G(v,v')$. By equating this to the scalar product, one obtains an isomorphism of TM_p and T^*M_p which is the classical operation of raising and lowering tensor sub- and superscripts. To see this, note that, for fixed v, $v' \to G(v,v')$ is a linear form on TM_p, and so a covector w, such that

$$G(v,v') = \langle w, v'\rangle \quad \text{for all} \quad v' \in TM_p. \tag{1.1.12}$$

As G is non-degenerate, the linear map $TM_p \ni v \to w \in T^*M_p$ is surjective and hence, as both spaces have the same dimension, it is also bijective. Defining w' in the same way, one sees that $(w,w') \to G(v,v')$ is a tensor $G^*(w,w')$ of type $(2,0)$, also symmetric and non-degenerate, and it is obvious that the inverse of the map defined by (1.1.12) is given by

$$G^*(w,w') = \langle w', v\rangle \quad \text{for all} \quad w' \in T^*M_p. \tag{1.1.13}$$

It is easy to see that, in local coordinates, the components of G^* are the matrix (g^{ij}) which is the inverse of (g_{ij}). Also, (1.1.12) and (1.1.13) are, in components,

$$\xi_i = g_{ij}\xi^j, \quad \xi^i = g^{ij}\xi_j \quad (i = 1, \ldots, h), \tag{1.1.14}$$

where the ξ^i are the components of v and the ξ_i are the components of w. These operations extend to tensors of all types.

From now on, we shall generally only consider manifolds with metrics, and treat (1.1.12) and (1.1.13) as an identification of TM_p and T^*M_p, which extends to the corresponding bundles, and fields. One thus has the classical notation, in which the ξ^i are 'contravariant components' and the ξ_i are the 'covariant components' of the same vector (or covector) v. This is convenient in calculations. Note that, with $\pi_* v = \xi$ and $\pi_* w = \eta$,

$$\langle v, w\rangle = \xi_i\eta^i = \xi^i\eta_i = g_{ij}\xi^i\eta^j = g^{ij}\xi_i\eta_j.$$

Also, in classical notation, $\langle v, v\rangle$ becomes the line element,

$$ds^2 = \langle dx, dx\rangle = g_{ij}(x)\,dx^i dx^j. \tag{1.1.15}$$

At this point, the affine connection, covariant differentiation, curvature tensor and other basic concepts of differential geometry, can be introduced in a coordinate-free way. However, as they will only be needed marginally in the sequel, we merely give a brief summary in terms of local coordinates and components.

The components of the affine connection determined by the metric are

$$\Gamma^i_{jk} = \tfrac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right). \tag{1.1.16}$$